FEATURES OF RIGHT TRIANGLES

The focus of this unit is on special attributes of right triangles. You will examine the geometric mean in proportions. and then examine the geometric mean in the relationship between a right triangle and its altitude. You will then look at the relationships of the sides of the 45-45-90 triangle and the 30-60-90 triangle. In the final part of the unit, you will examine one proof of the Pythagorean Theorem applying the geometric mean and a theorem from the unit.

Geometric Sequences and Geometric Mean

geometric sequence – A geometric sequence is a sequence found by multiplying the previous number by a given factor to determine the next successive number.

$$\{6, 12, 24, 48, ...\}$$

Notice that each number in the above sequence is twice as much as the number before it. Each successive term is determined by multiplying by a factor of 2.

x2 x2 x2

$$\{6, 12, 24, 48 ...\}$$

Example 1: Find the next three numbers in the squence shown above.

To find the next three numbers in the sequence, **follow the pattern**, multiplying by two each time.

x2 x2 x2

{6, 12, 24, 48**, 96, 192, 384...}**

The next three numbers in the sequence are 96, 192, and 384.

geometric mean - The geometric mean between two positive numbers, *a* and *b*, is the positive number x where $\frac{a}{x} = \frac{x}{b}$ or $x = \sqrt{ab}$.

$$\frac{a}{x} = \frac{x}{b}$$

$$x^2 = ab$$

$$x = \sqrt{ab}$$

The **geometric mean** of *a* and *b* is the square root of the product of *a* and *b*.

Example 2: Find the geometric mean between 6 and 14.

$a = 6$ $b = 14$

$$\frac{a}{x} = \frac{x}{b}$$

$$\frac{6}{x} = \frac{x}{14}$$

$$x^2 = 84$$

$$x = \sqrt{84}$$

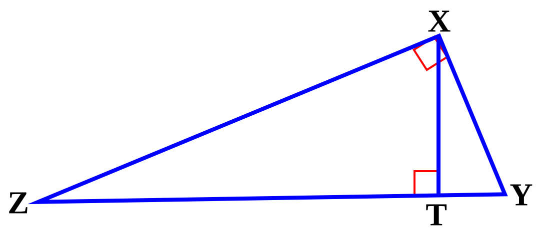
$x = 2\sqrt{21}$ or $x = 9.2$

Right Triangles and Altitudes

In geometry, the geometric mean is found in the relationship between right triangles and their altitudes.

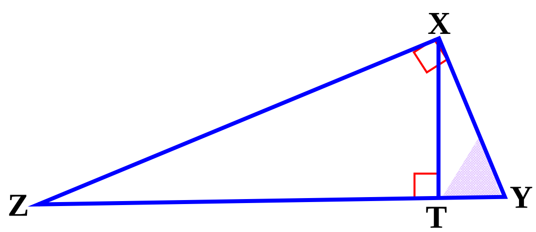
First, we'll examine right triangle *XYZ* with altitude $\overline{XT}$.

We can name three different triangles: $\triangle XYZ$, $\triangle XTY$, $\triangle XTZ$.



Is $\triangle XYZ \sim \triangle XTY$?

First, notice that the two triangles share $\angle Y$ and both triangles have a right angle.



Now, let's prove that $\triangle XYZ$ and $\triangle XTY$ are indeed similar.

Statement	Reason
$\angle ZXY$ is a right angle.	Definition of right triangle. and altitude.
$\angle XTY$ is a right angle.	Definition of the altitude of a triangle.
$\angle ZXY \cong \angle XTY$	Right angles are congruent.
$\angle Y \cong \angle Y$ (Angle Y is in both triangles.)	Reflexive Property
$\triangle ZXY \sim \triangle XTY$	AA Similarity Postulate

Take another look at the figure above and see if you can answer the following question.

Is $\triangle XYZ \sim \triangle XTZ$? Again, a proof can be developed to show that these two triangles are also similar based on the AA Similarity Postulate. The proof will be left to be determined in the problem set.

Now, let's consider the relationship between $\triangle XYZ$, $\triangle XTY$, and $\triangle XTZ$.

If $\triangle XTY \sim \triangle XYZ$ and $\triangle XYZ \sim \triangle XTZ$, then $\triangle XTY \sim \triangle XTZ$. (transitive property)

Therefore, $\triangle XYZ \sim \triangle XTY \sim \triangle XTZ$, and the following theorem can be written.

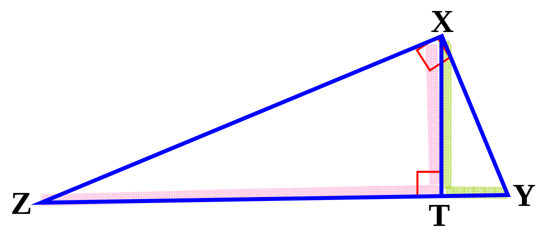
Theorem 20-A

In a right triangle, if an altitude is drawn from the vertex of the right angle to the hypotenuse, then the two triangles formed are similar to each other and to the given triangle.

Now, let's examine take a closer look at two of the similar triangles: $\triangle ZTX \sim \triangle XTY$

Notice that $\overline{ZT}$ and $\overline{XT}$ are the longer legs of the two right triangles and that $\overline{TX}$ and $\overline{TY}$ are the shorter legs. Thus, we can write the following proportion to compare the corresponding similar sides.

$$\frac{ZT}{TX} = \frac{TX}{TY}$$



Therefore, TX is the geometric mean between ZT and TY. The following theorem summarizes this idea.

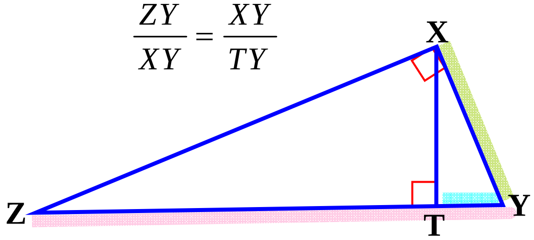
Theorem 20-B

In a right triangle, the measures of the altitude drawn from the vertex of the right angle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse created by the intersection of the hypotenuse and the altitude.

What other proportional statements can be made that involve a geometric mean?

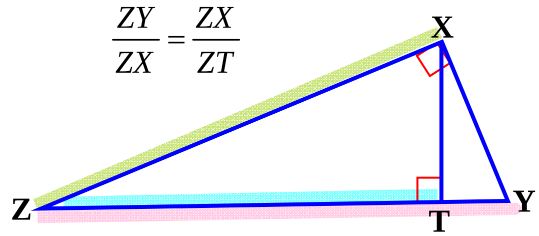
Look at $\triangle ZXY \sim \triangle XTY$. Write a proportional statement comparing the hypotenuses to the shorter legs of the right triangles.

$$\frac{ZY}{XY} = \frac{XY}{TY}$$



Look at $\triangle ZXY \sim \triangle ZTX$. Write a proportional statement comparing the hypotenuses to the longer legs of the right triangles.

$$\frac{ZY}{ZX} = \frac{ZX}{ZT}$$

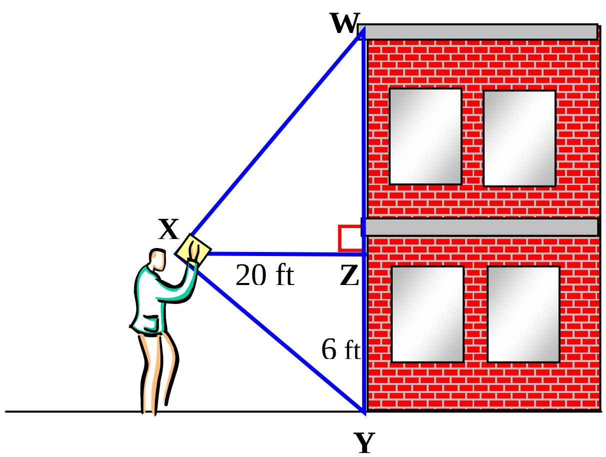


The following theorem summarizes the statements concluded above.

Theorem 20-C

In a right triangle with the altitude drawn to the hypotenuse, the measure of a leg is the geometric mean between the measure of the hypotenuse and the measure of the segment of the hypotenuse that is adjacent to the leg.

Example 3: Andrew must find the height of the city building. He used his book, which has a right angle at each of its four edges, to determine a right triangle from his line of site to the top and bottom of the building along the edges of the book. He was 20 feet away from the building and his eyes are 6 ft from the ground. How tall is the building?



$$\frac{\text{shorter segment of hypotenuse}}{\text{altitude}} = \frac{\text{altitude}}{\text{longer segment of hypotenuse}}$$ Theorem 20-B

Let x represent *WZ*.

$\frac{6}{20} = \frac{20}{x}$ Substitution

$6x = 400$ Cross products

$x \approx 66.7$ Division Property

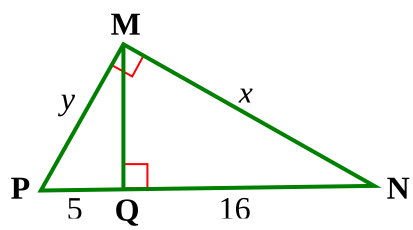
WZ is about 66.7 feet long.

Now, we can determine the height of the building by adding *WZ* and *ZY*.

$WZ + ZY = 66.7 + 6 = 72.7$ ft

The building is about 72.7 feet tall.

Example 4: What is the value of *x* and *y* in right triangle *MNP* with the given information shown below? .



MN (*x*) is a longer leg and *MP* (*y*) is a shorter leg of $\triangle MNP$. It will also be helpful to determine the length of the hypotenuse *PN* based on the given information.

PN = PQ + QN = 5 + 16 = 21.

We can make the following comparison of $\triangle MQP$ and $\triangle MNP$.

$$\frac{\text{shorter leg of } \triangle MQP}{\text{shorter leg of } \triangle MNP} = \frac{\text{hypotenuse of } \triangle MQP}{\text{hypotenuse of } \triangle MNP}$$ Theorem 20-C

$$\frac{5}{y} = \frac{y}{21}$$ Substitution

$$y^2 = 105$$ Cross Multiply

$$y = \sqrt{105}$$ Take the square root of each side.

$$y \approx 10.2$$

We can also make the following comparison of $\triangle MQP$ and $\triangle MNP$.

$\frac{\text{longer leg of } \triangle MQN}{\text{longer leg of } \triangle MNP} = \frac{\text{hypotenuse of } \triangle MQN}{\text{hypotenuse of } \triangle MNP}$	Theorem 20-C
$\frac{16}{x} = \frac{x}{21}$	Substitution
$x^2 = 336$	Cross Multiply
$x = \sqrt{336}$	Take the square root of each side.
$x \approx 18.3$	

Pythagorean Triples

Recall the Pythagorean Theorem.

Theorem 2-A Pythagorean Theorem	**In a right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.**

Theorem 20-D Converse of the Pythagorean Theorem	**If the sum of the squares of the measures of the two legs of a right triangle equals the square of the hypotenuse, then the triangle is a right triangle.**

Pythagorean triple – A Pythagorean triple is a set of three "whole" numbers that satisfies the Pythagorean Theorem where *c* is the greatest of the three whole numbers.

The whole numbers, 3, 4, and 5 are an example of a Pythagorean triple.

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

The wholes numbers 4, 5, and 6 are an example of three numbers that are NOT a Pythagorean triple.

$$a^2 + b^2 = c^2$$
$$4^2 + 5^2 \neq 6^2$$
$$16 + 25 \neq 36$$
$$41 \neq 36$$

Example 1: Are the lengths, 13, 84, and 85, lengths of the sides of a right triangle?

Since these three numbers are whole numbers, we will determine if they form a Pythagorean triple.

$a^2 + b^2 = c^2$

Does $13^2 + 84^2 \;=\; 85^2$?

Does $169 + 7056 \;=\; 7225$?

$7225 = 7225$ Yes!

The whole numbers 13, 84, and 85 forma a Pythagorean triple.

Here is an interesting formula to investigate:

> Suppose that *m* and n are two positive integers with *m* < *n*, then $n^2 - m^2$, $2mn$, and $n^2 + m^2$ is a **Pythagorean triple**.

Let's test this formula.

Example 2: If *n* = 5 and *m* = 3, what Pythagorean triple will these numbers produce?

$n^2 - m^2$	$2mn$	$n^2 + m^2$
$5^2 - 3^2$	2(5)(3)	$5^2 + 3^2$
16	30	34

Check: Does $16^2 + 30^2 = 34^2$?

$16^2 + 30^2 = 34^2$

$256 + 900 = 1156$

$1156 = 1156$

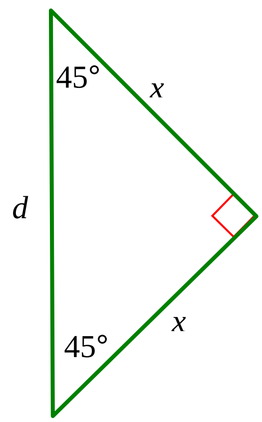
Yes!

45-45-90 Right Triangle

In a 45-45-90 degree triangle, the length of the hypotenuse can be determined by multiplying $\sqrt{2}$ times the leg.

Let's take a look at why this is so!

When given a right isosceles triangle, the angles opposite the congruent sides are congruent 45 degree angles. Determine the length of *d* for any isosceles right triangle with hypotenuse *d* and leg *x*.



We will use the Pythagorean Theorem to develop the formula for finding the hypotenuse (*d*) when given the length of either of the congruent sides.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$x^2 + x^2 = d^2$	Substitute $(a = x, b = x, c = d)$
$2x^2 = d^2$	Collect like terms.
$\sqrt{2x^2} = \sqrt{d^2}$	Take the square root of each side.
$\sqrt{2}\sqrt{x^2} = \sqrt{d^2}$	Split the radical.
$d = x\sqrt{2}$	

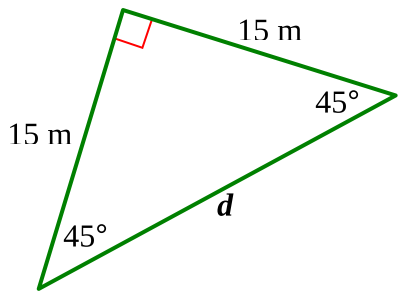
To summarize the relationship of the lengths of the three sides of a 45-45-90 degree triangle:

leg a = leg b	$\rightarrow$	x
hypotenuse	$\rightarrow$	$x\sqrt{2}$

Theorem 20-E

In a 45-45-90 degree right triangle, the length of the hypotenuse can be determined by multiplying $\sqrt{2}$ times the leg.

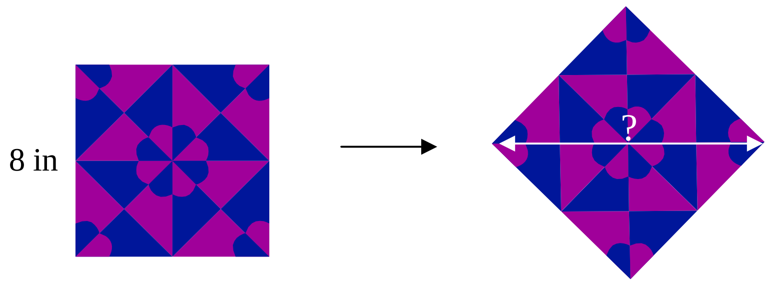
Example 1: Find the hypotenuse of a right isosceles triangle with a leg measuring 15 meters.



$d = x\sqrt{2}$	Theorem 20-E
$d = 15\sqrt{2}$	Substitution
$d \approx 21.2$	Simplify

The hypotenuse measures approximately 21.2 meters.

Example 2: A quilter wants to place her block, an 8-inch square, on "point" in her quilt. How wide will the block be "on point"?



In this problem, we must figure the length of the diagonal of the square. Recall that the diagonal of a square divides the square into two isosceles right triangles, that is, two 45-45-90 right triangles.

To solve, multiply the length of the side times the square root of 2.

$d = x\sqrt{2}$	Theorem 20-E
$d = 8\sqrt{2}$	Substitution
$d \approx 11.3$	Simplify

The width of the block "on point" is approximately 11.3 inches.

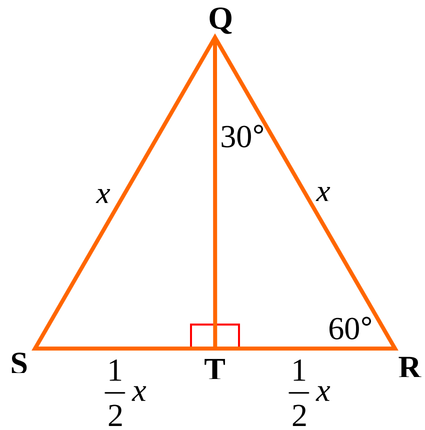
30-60-90 Right Triangle

In a 30-60-90 degree triangle, the length of the hypotenuse is twice as long as the shorter leg and the longer leg equals the shorter leg multiplied by $\sqrt{3}$.

Let's take a look at why this is so!

First we can start with an equilateral triangle and draw its altitude.

The altitude of an equilateral triangle divides it into two 30-60-90 degree triangles.



*Notice that in either of the smaller triangles, the shorter leg is opposite a 30 degree angle and is half of the length of the of one side of the equilateral triangle.

$$\text{length of shorter leg } = \frac{1}{2}(\text{length of hypotenuse})$$

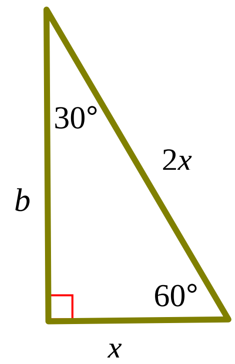
OR

$$2(\text{length of shorter leg}) = 2\left(\frac{1}{2}(\text{length of hypotenuse})\right)$$

EQUALS

$$2(\text{length of shorter leg}) = \text{length of hypotenuse}$$

Thus, if x equals the length of the shorter leg, then the length of the hypotenuse is $2x$.



We can then use the Pythagorean Theorem to develop a formula for finding the length of the side that is opposite the 60 degree angle.

Let x represent the length of the shorter leg and b represent the length of the longer leg.

Thus, the length of the hypotenuse equals $2x$.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$x^2 + b^2 = (2x)^2$	Substitute $(a = x, c = 2x)$
$x^2 + b^2 = 4x^2$	Square $2x$ $(2x \cdot 2x = 4x^2)$
$b^2 = 3x^2$	Subtract and collect like terms.
$\sqrt{b^2} = \sqrt{3x^2}$	Take the square root of each side.
$\sqrt{b^2} = \sqrt{3}\sqrt{x^2}$	Separate the radical.
$b = x\sqrt{3}$	Simplify

Now, we can use $x\sqrt{3}$ to represent the length of the longer leg.

To summarize the relationship of the three sides of a 30-60-90 degree triangle:

shorter leg	$\rightarrow$	x
longer leg	$\rightarrow$	$x\sqrt{3}$
hypotenuse	$\rightarrow$	$2x$

Theorem 20-F

In a 30-60-90 degree right triangle, the length of the hypotenuse is twice as long as the shorter leg, and the longer leg equals the shorter leg multiplied by $\sqrt{3}$.

Example 1: What are the measures of the legs of a 30-60-90 degree triangle with a hypotenuse that measures 14 feet?

Step 1: Given: hypotenuse = 14, therefore $2x = 14$

Step 2: shorter leg = x

$2x = 14$ length of hypotenuse

$x = 7$ Divide by 2

The shorter leg = 7 feet.

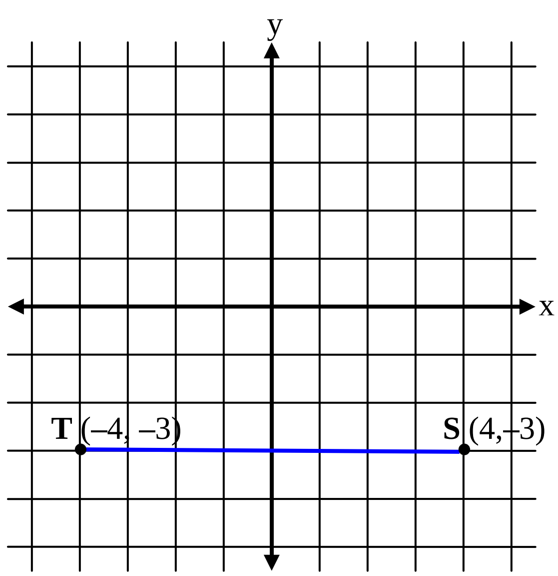
Step 3: longer leg = $x\sqrt{3}$

$x\sqrt{3}$ Length of longer leg

$7\sqrt{3}$ Substitute $(x = 7)$

The longer leg = $7\sqrt{3}$ feet.

Example 2: Triangle *STU* is a 30-60-90 triangle with right angle *T*. Angle *S* measures 30 degrees. Find the coordinates for vertex *U* given vertices $T(-4, -3)$ and $S(4, -3)$.



To begin, we know that $\angle S$ measures $30°$. Thus $\angle U$ must measure $60°$ since $\angle T$ is a right angle.

Now, we can compare the sides.

shorter leg $\overline{TU}$	opposite $\angle S$ (30-degree angle) and perpendicular to $\overline{TS}$
longer leg $\overline{TS}$	opposite $\angle U$ (60-degree angle) and perpendicular to $\overline{TU}$
hypotenuse $\overline{US}$	opposite $\angle T$ (90-degree angle)

First, we will find the length of *TS* by using the distance formula.

$T(-4,-3)$ $\quad S(4,-3)$

$TS = \sqrt{[4-(-4)]^2+[-3-(-3)]^2}$	Distance formula
$TS = \sqrt{64+0}$	Simplify
$TS = 8$	Simplify

Thus,

shorter leg $\overline{TU}$	opposite $\angle S$ (30-degree angle) and perpendicular to $\overline{TS}$
longer leg $\overline{TS} = 8$	opposite $\angle U$ (60-degree angle) and perpendicular to $\overline{TU}$
hypotenuse $\overline{US}$	opposite $\angle T$ (90-degree angle)

Next, we will find the length of *TU*. We will use the fact that the longer leg of a 30-60-90 triangle equals $(\sqrt{3})x$ when *x* represents the shorter leg.

$(\sqrt{3})x = 8$ — Theorem 20-F

$x = \frac{8}{\sqrt{3}}$ — Divide both sides by $\sqrt{3}$.

$x = \frac{8}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$ — Rationalize the denominator.

$x \approx 4.6$ — Simplify

Thus,

shorter leg $\overline{TU} = 4.6$ — opposite $\angle S$ (30-degree angle) and perpendicular to $\overline{TS}$

longer leg $\overline{TS} = 8$ — opposite $\angle U$ (60-degree angle) and perpendicular to $\overline{TU}$

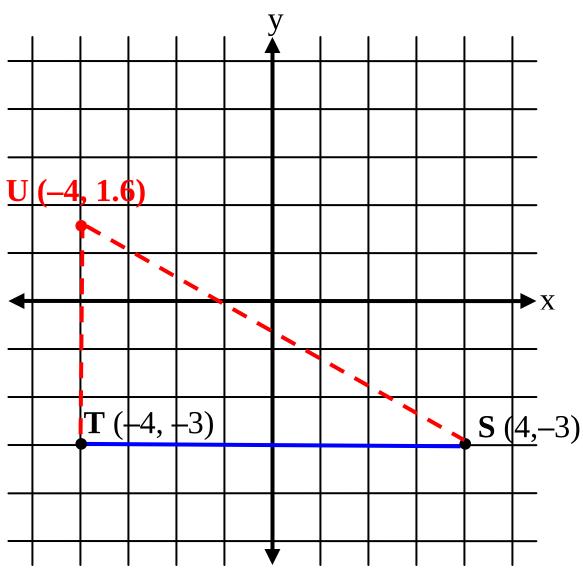
hypotenuse $\overline{US}$ — opposite $\angle T$ (90-degree angle)

Finally, we have enough information to determine the coordinates of vertex *U*.

Since *TU* is perpendicular to TS, the *x*-coordinate of point *U* is –4. The *y*-coordinate of point *U* can be determined by adding the length of *TU* to the *y*-coordinate of point T.

y-coordinate = –3 + 4.6 or 1.6.

The coordinates of point U are (–4, 1.6).



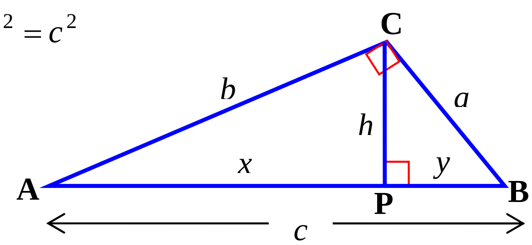
One Proof of the Pythagorean Theorem

In the activity below, a series of questions are asked to guide you through one proof of the Pythagorean Theorem. Work through the proof and then compare your answers to the ones provided at the end of the activity.

Theorem 2-A Pythagorean Theorem	**In a right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.**

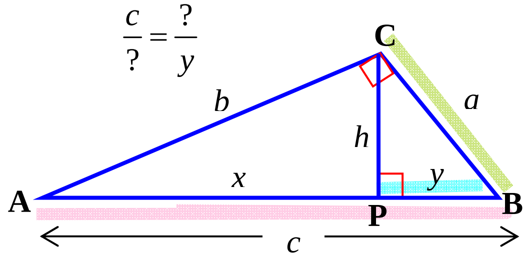
Given: $\triangle ABC$ with right angle C and altitude h drawn from the right angle to the hypotenuse.

Prove: $a^2 + b^2 = c^2$



(a) Apply Theorem 20-C and fill in the missing information in the proportion shown below.

$$\frac{c}{?} = \frac{?}{y}$$

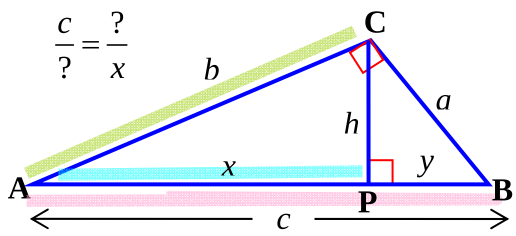


(b) Refer to the proportion and use cross products to determine the missing part of the equation given below.

$$? = cy$$

(c) Apply Theorem 20-C and fill in the missing information in the proportion shown below.

$$\frac{c}{?} = \frac{?}{x}$$



(d) Refer to the proportion and use cross products to determine the missing part of the equation given below.

$$? = cx$$

(e) Use the information derived in the previous steps and substitution to determine the missing parts in the equation shown below.

$$? \; + \; ? = cy + cx$$

(f) Factor the right side of the previous equation, and then state the missing part.

$$a^2 + b^2 = ?(y + x)$$

(g) Look at the original diagram and then fill in the missing part.

$$c = x + ?$$

(h) Refer back to the previous two steps and use substitution to fill in the missing part in the equation below.

$$a^2 + b^2 = c(?)$$

(i) Simply and the proof of the Pythagorean Theorem is complete.

$$a^2 + b^2 = ?$$

Answers

a.) a, a	b.) a^2	c.) b, b	d.) b^2	e.) a^2, b^2
f.) c	g.) y	h.) c	i.) c^2	