RATIO, PROPORTION, AND SIMILAR FIGURES

In this unit you will review solving problems with ratios and proportions applying "Equality of Cross Products". You will then examine similar figures and scale factors. You will then take a closer look at postulates and theorems that are used to determine the similarity of two triangles.

Ratios and Proportions

Similar Polygons

Similar Triangles

Simplifying Radicals (*Review*)

Ratios and Proportions

ratio – A ratio is a comparison of two quantities. The ratio of a to b can be expressed as $\frac{a}{b}$, where b is not equal to zero. Another way to write a ratio is $a:b$.

Example 1: Sebastian is a friendly and lovable extra large Rottweiler. His height in the picture is $1\frac{1}{8}$ inches. His actual height is 36 inches. What is the ratio of his height in the picture to his actual height?

$$\frac{a}{b}=\frac{1\frac{1}{8}}{36}=1\frac{1}{8}\div 36=\frac{9}{8}\times\frac{1}{36}=\frac{9}{8}\times\frac{1}{36}=\frac{1}{32}$$

The ratio of comparing the two heights simplifies to $\frac{1}{32}$ or 1:32.

unit ratio – A unit ratio is a ratio simplified to a denominator of one.

Example 2: Simplify $\frac{48}{10}$ to a unit ratio.

$$\frac{48}{10}\div\frac{10}{10}=\frac{4.8}{1}$$

$\frac{48}{10}$ simplifies to $\frac{4.8}{1}$ or $(4.8):1$.

proportion – A proportion is an equation which states that two ratios are equal.

Equality of Cross Products

For any real numbers, a, b, c, and d, where b and d are not equal to zero,

$\frac{a}{b} = \frac{c}{d}$ if and only if, $ad = bc$.

Now let's consider a proportion stated in general terms to examine the "Equality of Cross Products".

$\frac{a}{b} = \frac{c}{d}$	
$b \cdot \frac{a}{b} = b \cdot \frac{c}{d}$	Multiply each side by b.
$a = \frac{bc}{d}$	Simplify
$d \cdot a = d \cdot \frac{bc}{d}$	Multiply each side by d.
$da = bc$	Simplify
$ad = bc$	Commutative Property of Multiplication
$\frac{a}{b} = \frac{c}{d}$	$ad = bc$ (Cross Products)

Example 3: Solve $\frac{7x-8}{3}=\frac{5}{6}$.

Apply the "Equality of Cross Products" to solve.

$$\begin{aligned} ad &= bc \\ (7x-8)(6) &= 3(5) \\ 42x-48 &= 15 \\ 42x &= 63 \\ x &= \frac{63}{42} \text{ or } \frac{3}{2} \end{aligned}$$

Example 4: Cindy surveyed her class and found that 11 out of 25 played video games on their own videogame box. There are 500 students in the Cindy's class. Based on Cindy's survey results, predict about how many students in her class may have their own videogame box.

Solve by stating two ratios comparing "part to whole of the group surveyed" and "part of the class to whole class".

x = number of students that may have a videogame box

$$\frac{11}{25}=\frac{x}{500}$$

$$\begin{aligned} 11(500) &= 25(x) \\ 5500 &= 25x \\ 220 &= x \end{aligned}$$

Approximately 220 students in Cindy's class may have a videogame box based on Cindy's survey sample.

*Note: Several samples should be taken to get a better estimate of the actual number of students who have a videogame box.

Example 5: Three angles are supplementary. The ratio of their measures is $2:5:8$. Find the measure of each angle.

x = one part of the total 180 degree angle.

Then, $2x$, $5x$, and $8x$ represents the three angles.

Write the equation to represent the total of the three angles.

$$2x + 5x + 8x = 180$$

$$15x = 180 \qquad \text{Simplify}$$

$$x = 12 \qquad \text{Divide}$$

First angle: $2x = 2(12) = 24°$
Second angle: $5x = 5(12) = 60°$
Third angle: $8x = 8(12) = 96°$

Check: $24° + 60° + 96° = 180°$
$180° = 180°$

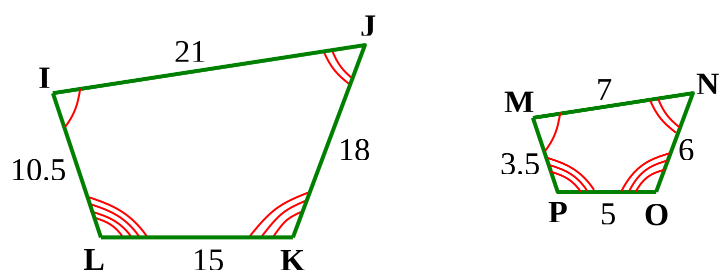
Similar Polygons

similar figures – Similar figures are figures that have the same shape but are different in size.

similar polygons – Similar polygons are polygons that have congruent corresponding angles and the measures of their corresponding sides are proportional.

Quadrilateral *IJKL* and *MNOP* are similar. The "tilde" symbol (~) represents "is similar to".

Quadrilateral *IJKL* ~ Quadrilateral *MNOP*



Therefore, the congruent angles are:

$\angle I \cong \angle M \qquad \angle J \cong \angle N \qquad \angle K \cong \angle O \qquad \angle L \cong \angle P$

…and the corresponding proportional sides are:

$$\frac{IJ}{MN} = \frac{JK}{NO} = \frac{LK}{PO} = \frac{IL}{MP}$$

scale factor – The scale factor for two similar polygons is the ratio of the lengths of any two corresponding sides.

In quadrilateral *IJKL* and quadrilateral *MNOP* shown above, all of the proportional sides can be simplified to the same scale factor, $\frac{3}{1}$.

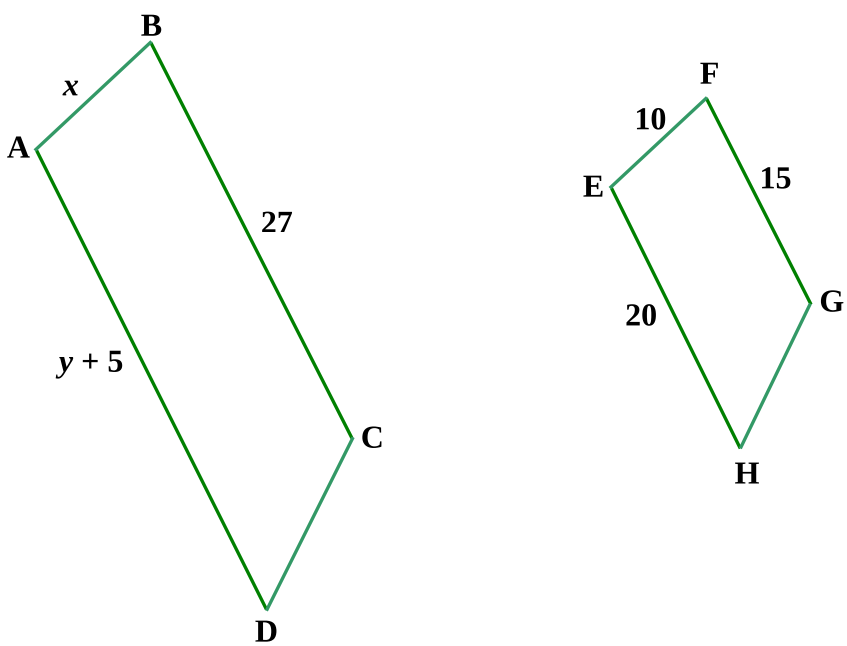
$$\frac{IJ}{MN} \quad = \quad \frac{JK}{NO} \quad = \quad \frac{LK}{PO} \quad = \quad \frac{IL}{MP}$$

$$\frac{21}{7} = \frac{3}{1} \qquad \frac{18}{6} = \frac{3}{1} \qquad \frac{15}{5} = \frac{3}{1} \qquad \frac{10.5}{3.5} = \frac{3}{1}$$

*Note: We can also state that the scale factor of quadrilateral *MNOP* to quadrilateral *IJKL* is $\frac{1}{3}$.

Refer to the diagram below to solve the first three examples.

Given: Quadrilateral *ABCD* ~ Quadrilateral *EFGH*



Example 1: What is the scale factor of quadrilateral *ABCD* to quadrilateral *EFGH*?

We will use corresponding sides BC to FG to write the ratio.

$$\frac{BC}{FG} = \frac{27}{15} = \frac{9}{5}$$

The scale factor is $\frac{9}{5}$.

*Note: Simplify fractions to lowest terms but write them as fractions (not mixed fractions or decimals).

Example 2: Find the value of x.

Write a proportion using values of corresponding sides and the variable x.

$$\frac{AB}{EF} = \frac{BC}{FG}$$

$$\begin{aligned} \frac{x}{10} &= \frac{27}{15} \\ x(15) &= 10(27) \\ 15x &= 270 \\ x &= 18 \end{aligned}$$

Check: $\frac{18}{10} = \frac{9}{5}$ which is the scale factor.

Example 3: Find the value of y.

$$\frac{AD}{EH} = \frac{BC}{FG}$$

$$\begin{aligned} \frac{y+5}{20} &= \frac{27}{15} \\ (y+5)(15) &= 20(27) \\ 15y + 75 &= 540 \\ 15y &= 465 \\ y &= 31 \end{aligned}$$

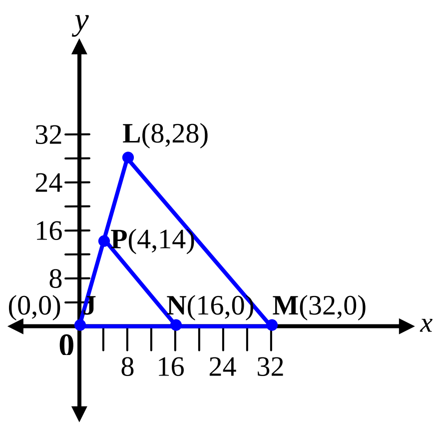
Check: $\frac{31+5}{20} = \frac{36}{20} = \frac{9}{5}$ which is the scale factor.

Example 4: Triangle *JNP* has vertices J(0,0), N(16,0), and P(4,14). Multiply each of the vertices' coordinates by 2 to create triangle *JML*. Is triangle *JML* similar to triangle *JNP*?

Step 1: Find the new coordinates.

Coordinates	New Coordinates
J (0, 0)	J [2(0), 2(0)] = (0,0)
N(16, 0)	M [2(16), 2(0)] = (32,0)
P (4, 14)	L [2(4), 2(14)] = L(8, 28)

Step 2: Graph both triangles.



Recall that similar polygons are polygons that have congruent corresponding angles and the measures of their corresponding sides are proportional.

First we'll check for congruent corresponding angles.

Step 3: To determine if the angles are congruent for both triangles, we will check to see if *PN* P*LM* . If so, then we can apply the properties of parallel lines to determine congruence of the triangles' angles.

What is the slope of $\overline{PN}$? What is the slope of $\overline{LM}$?

$$\text{Slope of } \overline{PN} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14-0}{4-16} = \frac{14}{-12} = -\frac{7}{6}$$

$$\text{Slope of } \overline{LM} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{28-0}{8-32} = \frac{28}{-24} = -\frac{7}{6}$$

Since the slopes are equal, $\overline{PN}$ P $\overline{LM}$

Therefore:

$\angle JPN \cong \angle JLM$ Corresponding angles of parallel lines cut by a transversal are congruent.

$\angle JNP \cong \angle JML$ Corresponding angles of parallel lines cut by a transversal are congruent.

$\angle J \cong \angle J$ Reflexive Property.

Thus, the corresponding angles are congruent.

Now we will check for a proportional relationship between the sides.

Step 4: We will now find the length of the sides of each triangle.

Triangle JNP J(0, 0), N(16, 0), P(4, 14)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$JN = \sqrt{(16-0)^2 + (0-0)^2} = \sqrt{256} = 16$$

$$JP = \sqrt{(4-0)^2 + (14-0)^2} = \sqrt{212} = 2\sqrt{53}$$

$$NP = \sqrt{(4-16)^2 + (14-0)^2} = \sqrt{340} = 2\sqrt{85}$$

Triangle JML **J(0,0), M(32,0), L(8,28)**

$$JM = \sqrt{(32-0)^2+(0-0)^2} = \sqrt{1024} = 32$$

$$JL = \sqrt{(8-0)^2+(28-0)^2} = \sqrt{848} = 4\sqrt{53}$$

$$ML = \sqrt{(8-32)^2+(28-0)^2} = \sqrt{1360} = 4\sqrt{85}$$

Check the ratios of the corresponding sides.

$$\frac{JN}{JM} = \frac{16}{32} = \frac{1}{2}$$

$$\frac{JP}{JL} = \frac{2\sqrt{\cancel{53}}}{4\sqrt{\cancel{53}}} = \frac{1}{2}$$

$$\frac{NP}{ML} = \frac{2\sqrt{\cancel{85}}}{4\sqrt{\cancel{85}}} = \frac{1}{2}$$

Therefore: $\dfrac{JN}{JM} = \dfrac{JP}{JL} = \dfrac{NP}{ML} = \dfrac{1}{2}$

***Conclusion*: Since both the corresponding angles are congruent and the corresponding sides are proportional, triangle *JNP* is similar to triangle *JML*.**

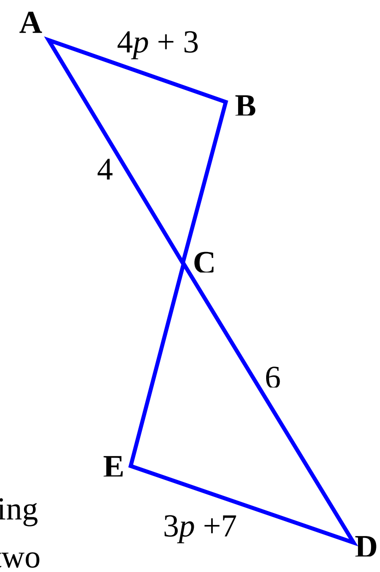
Similar Triangles

We will now examine postulates and theorems that apply to similar triangles.

Postulate 16-A **AA Similarity**	**If two angles of one triangle are congruent to two angles of a second triangle, then the triangles are similar.**

Refer to the diagram below to solve the first three examples.

Given: $AB \parallel ED$



Example 1: Is $\triangle ABC$ similar to $\triangle DEC$?

$\angle BCA$ and $\angle ECD$ are vertical angles and congruent.

$\angle A$ and $\angle D$ are alternate interior angles of parallel lines $\overline{AB}$ and $\overline{ED}$ cut by transversal $\overline{AD}$; therefore, they are congruent.

Thus, $\triangle ABC \sim \triangle EDC$ by the AA Similarity Postulate.

*Note: $\angle B$ and $\angle E$ are also alternate interior angles of parallel lines and therefore congruent; but, by the AA Similarity Postulate, only two pairs of corresponding angles must be shown to be congruent to deduce that two triangles are similar.

Example 2: Find the value of *p*.

Since $\triangle ABC \sim \triangle EDC$, we can write a proportion based on the definition of similar polygons.

$$\frac{AB}{DE} = \frac{AC}{DC}$$

$$\frac{4p+3}{3p+7} = \frac{4}{6}$$

$$(4p+3)6 = (3p+7)4$$

$$24p+18 = 12p+28$$

$$12p = 10$$

$$p = \frac{10}{12} = \frac{5}{6}$$

The value of p is $\frac{5}{6}$.

Example 3: Find AB and DE.

Use substitution to determine the length of each segment.

$$AB = 4p + 3 = 4(\frac{5}{6}) + 3 = 3\frac{1}{3} + 3 = 6\frac{1}{3}$$

$$DE = 3p + 7 = 3(\frac{5}{6}) + 7 = 2\frac{1}{2} + 7 = 9\frac{1}{2}$$

Check: $\frac{AC}{DC} = \frac{4}{6} = \frac{2}{3}$ (scale factor)

$$\frac{AB}{DE} = \frac{6\frac{1}{3}}{9\frac{1}{2}} = 6\frac{1}{3} \div 9\frac{1}{2} = \frac{19}{3} \cdot \frac{2}{19} = \frac{2}{3} \text{ (scale factor)}$$

The proportional sides, $AB : DE$, equal the same scale factor, $2:3$.

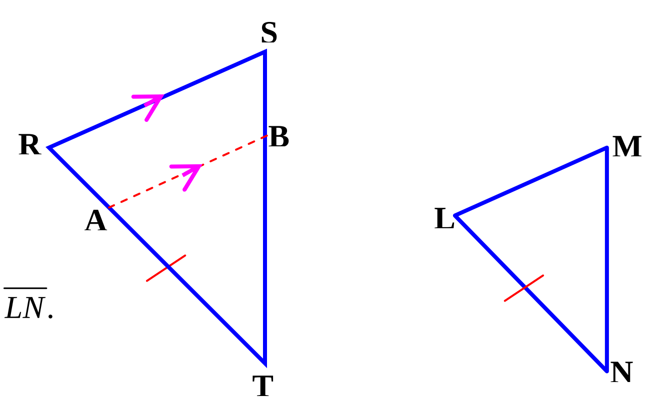
Theorem 16-A
SSS Similarity

If the measure of the corresponding sides of two triangles is proportional, then the triangles are similar.

Given:

$$\frac{LM}{RS} = \frac{MN}{ST} = \frac{LN}{RT}$$

Prove: $VLMN \sim VRST$



Locate point A on $\overline{RT}$ so that $\overline{AT} \cong \overline{LN}$.

Draw $\overline{AB}$ parallel to $\overline{RS}$.

1. $\overline{AB}\,\mathrm{P}\,\overline{RS}$	Given
2. $\angle TAB \cong \angle R,\ \angle TBA \cong \angle S$	Corresponding angles of parallel lines cut by a transversal are congruent.
3. $\mathrm{V}TAB \sim \mathrm{V}TRS$	AA Similarity Postulate
4. $\dfrac{TA}{TR} = \dfrac{TB}{TS} = \dfrac{AB}{RS}$	Definition of similar polygons
5. $\overline{LN} \cong \overline{AT}$	Given (auxiliary segment added in)
6. $\dfrac{LN}{TR} = \dfrac{TB}{TS} = \dfrac{AB}{RS}$	Substitution
7. $\dfrac{LN}{TR} = \dfrac{MN}{TS} = \dfrac{LM}{RS}$	Given
8. If $\dfrac{TB}{TS} = \dfrac{LN}{TR}$ and $\dfrac{LN}{TR} = \dfrac{MN}{TS}$, then $\dfrac{TB}{TS} = \dfrac{MN}{TS}$.	Transitive Property
9. $(TB)(TS) = (TS)(MN)$	Cross Products
10. $TB = MN$	Division
11. $\overline{TB} \cong \overline{MN}$	Definition of congruence
12. If $\dfrac{AB}{RS} = \dfrac{LN}{TR}$ and $\dfrac{LN}{TR} = \dfrac{LM}{RS}$, then $\dfrac{AB}{RS} = \dfrac{LM}{RS}$	Transitive Property
13. $(AB)(RS) = (RS)(LM)$	Cross Products
14. $AB = LM$	Division
15. $\overline{AB} \cong \overline{LM}$	Definition of congruence
16. $\therefore \mathrm{V}LMN \cong \mathrm{V}ABT$	SSS Postulate ($\overline{LN} \cong \overline{AT},\ \overline{MN} \cong \overline{TB},\ \overline{AB} \cong \overline{LM}$)
17. $\angle N \cong \angle T$	CPCTC
18. $\angle TBA \cong \angle M$	CPCTC
19. $\angle TBA \cong \angle S$	(from step 3)
20. If $\angle M \cong \angle TBA$ and $\angle TBA \cong \angle S$, then $\angle M \cong \angle S$.	Transitive Propety
22. $\therefore \mathrm{V}LMN \sim \mathrm{V}RST$	AA Similarty Postulate

Theorem 16-B
SAS Similarity

If the measures of two sides of a triangle are proportional to the measures of two corresponding sides of a second triangle, and the included angles are congruent, then the triangles are similar.

Example 4: Based on the figure and information given below, name the two similar triangles and give reasons for your conclusions.

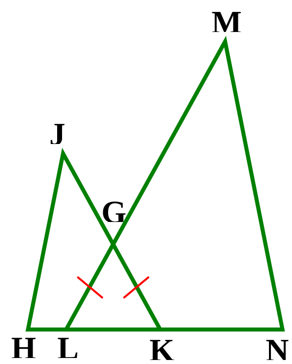
Given: $\overline{LG} \cong \overline{KG}$

$KJ = 12$

$LN = 15$

$HK = 10$

$ML = 18$



$\angle HKJ \cong \angle NLM$ Angles opposite congruent sides of a triangle are congruent.

Look at the numbers of the given sides to make proportional relationships.

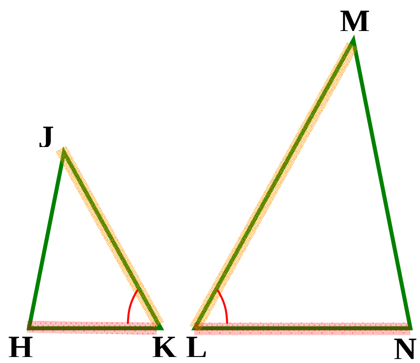
$$\frac{KJ}{ML} = \frac{12}{18} = \frac{2}{3} \qquad \frac{HK}{LN} = \frac{10}{15} = \frac{2}{3}$$

If $\frac{KJ}{ML} = \frac{2}{3}$ and $\frac{2}{3} = \frac{HK}{LN}$, then $\frac{KJ}{ML} = \frac{HK}{LN}$ Transitive Property

We have shown that two sides are proportional and an included angle is congruent for triangles HJK and NML.

Therefore, $\text{V}HJK \sim \text{V}NML$ by the SAS Similarity Theorem

In the figure below, the triangles are shown as non-overlapping triangles. The proportional sides and included angle are marked and highlighted.



We must mention one more theorem about similar triangles.

Theorem 16-C	**The similarity of triangles is reflexive, symmetric, and transitive.**

Simplifying Radicals

Sometimes it will be necessary to simplify radicals to produce like radicands. A square root is in simplest form when:

- There are no perfect square factors of the radicand.
- The radicand is not a fraction.
- No radical is in the denominator.

To simplify a radical

a.) Look for a perfect square factor.

Perfect squares are numbers that are squares of integers. Perfect squares can be calculated as follows: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, and so on.

The first 15 perfect squares are listed below:

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225...}

b.) Factor the radicand using the perfect square.

c.) Leave any factors that are not perfect squares under the radical.

Example #1: Simplify $\sqrt{18}$

a.) Eighteen contains a perfect square factor of 9.

b.) Factor the radicand using the perfect square of 9.

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2}$$

c.) Write $\sqrt{9}$ as 3 and simplify.

$$\sqrt{18} = 3 \times \sqrt{2}$$

$$\sqrt{18} = 3\sqrt{2}$$

Example #2: Simplify $\sqrt{275}$

$$\sqrt{275} = \sqrt{25 \cdot 11}$$

$$= \sqrt{25} \cdot \sqrt{11}$$

$$= 5 \cdot \sqrt{11}$$

$$= 5\sqrt{11}$$