SPECIAL PARALLELOGRAMS AND QUADRILATERALS

In this unit you will examine special parallelograms and quadrilaterals. You will examine the similarities and differences of rhombi and squares. You will then explore quadrilaterals that are not parallelograms. You will take a close look at trapezoids and be introduced to kites.

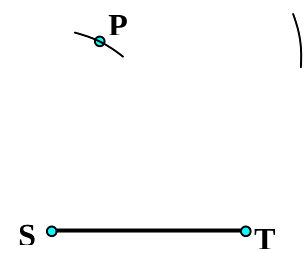
Rhombi and Squares

rhombus – A rhombus is a parallelogram with all four sides congruent (*rhombi* is the plural of rhombus).

First we'll take a look at how to construct a rhombus.

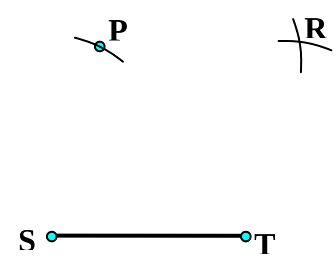
Step 1: Draw $\overline{ST}$.

Step 2: Place the metal point of the compass at point S and adjust the compass so that the pencil point touches point P.

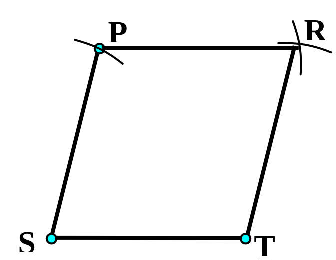


Step 3: Keeping the compass setting the same and the metal point on S, draw an arc above $\overline{ST}$. Label any point on the arc as point P.

Step 4: Move the metal point of the compass to point P. Without changing the setting of the compass, draw an arc to the right of point P.

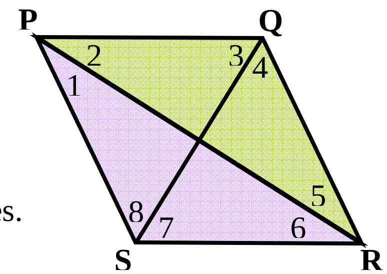


Step 5: Move the metal point of the compass to point *T*. Without changing the setting of the compass, draw an arc above $\overline{ST}$ that intersects with the arc drawn from point P. Label the point of intersection, point R.



Step 6: Use a ruler or straightedge to draw $\overline{SP}, \overline{TR},$ and $\overline{PR}$.

Theorem 15-A **The diagonals of a rhombus bisect its four angles.**



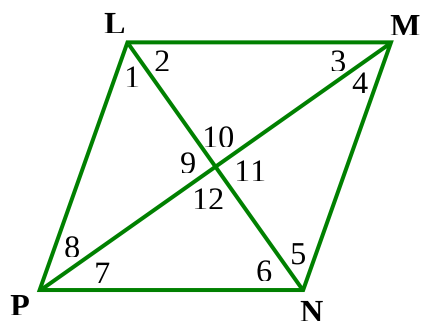
Given: Rhombus *PQRS*

Prove: Each diagonal bisects a pair of opposite angles.

Statements	Reasons
1. Rhombus *PQRS*	Given
2. $\overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{PS}$	Definition of rhombus
3. $\overline{PR} \cong \overline{PR}$	Reflexive Property
4. $\triangle PQR \cong \triangle PSR$	SSS
5. $\angle 1 \cong \angle 2$	CPCTC
6. $\angle 5 \cong \angle 6$	CPCTC
7. Diagonal *PR* bisects angles *SPQ* and *SRQ*.	Definition of angle bisector

What about diagonal QS? Does it bisect angles PQR and PSR? You will complete the proof in the problem set.

Example 1: In rhombus *LMNP*, find the measures of angles 7 and 8 if $m\angle 7 = 5x + 4$ and $m\angle 8 = 9x - 20$.



Since the diagonals of a rhombus bisect its angles, the measurements of angles 7 and 8 are equal.

$$m\angle 7 = m\angle 8$$
$$5x + 4 = 9x - 20$$
$$24 = 4x$$
$$6 = x$$

$$m\angle 7 = 5x + 4 = 5(6) + 4 = 34°$$

The measurement of angle 8 should also equal $34°$. We will check by substitution.

$$m\angle 8 = 9x - 20 = 9(6) - 20 = 34°$$

Therefore, $m\angle 7 = m\angle 8 = 34°$.

Theorem 15-B

The diagonals of a rhombus are perpendicular.

Example 2: In the previous example, if $m\angle 12 = x^2 + 15$, what is the value of *x*?

Since the diagonals of a rhombus are perpendicular, angles 9, 10, 11, and 12 each measure $90°$; thus, they are equal.

$$m\angle 9 = m\angle 12$$
$$x^2 + 15 = 90$$
$$x^2 = 75$$
$$x = \sqrt{75}$$
$$x \approx 8.66$$

Theorem 15-C

If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

Now let’s take a look at a special rhombus, the square!

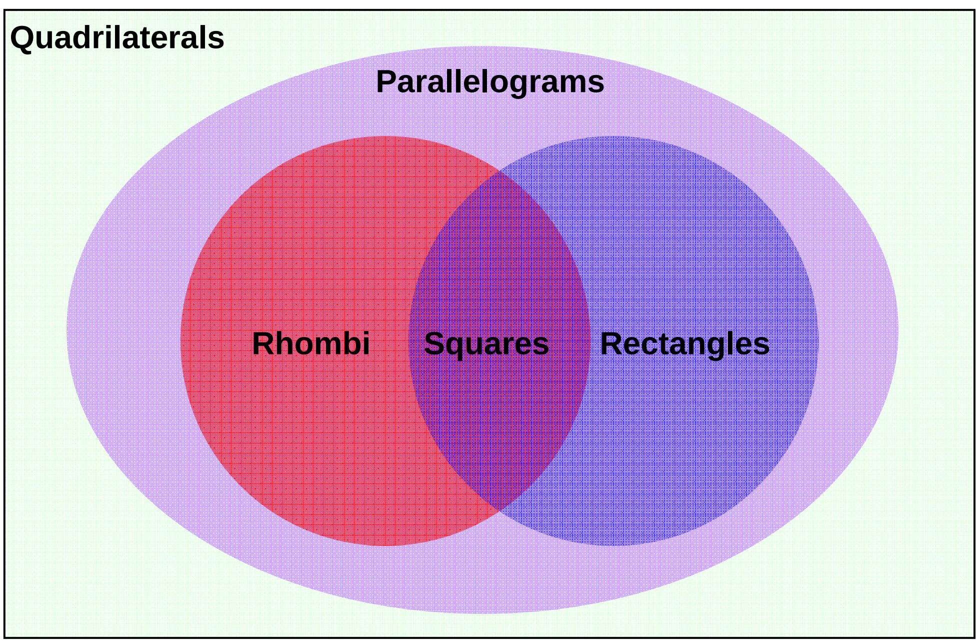
square – A square is a quadrilateral with four right angles and four congruent sides. If a quadrilateral is both a rhombus and a rectangle, then it is a square.

The Venn diagram below shows how the special quadrilaterals are related.

Notice that the region that represents the squares overlaps the regions that represent the rhombi and rectangles. This shows that all squares are rhombi and all squares are also rectangles.

Notice that the regions representing the rhombi and the rectangles are located within the region that represents parallelograms. This means that all rhombi and all rectangles are parallelograms.

Finally, notice that the region that represents the parallelograms falls within the region that represents the quadrilaterals. This means that all parallelograms are quadrilaterals. The area outside of the parallelograms (the white area) represents all other quadrilaterals.



In set notation, we can say that the parallelograms are a subset of the quadrilaterals, the rhombi are a subset of the parallelograms, the rectangles are a subset of the parallelograms, and the squares are a subset of both the rhombi and the rectangles.

squares $\subset$ rectangles $\subset$ parallelograms $\subset$ quadrilaterals

squares $\subset$ rhombi $\subset$ parallelograms $\subset$ quadrilaterals

We can state the relationships another way. We can say:

"All parallelograms are quadrilaterals."

"All rhombi are parallelograms."

"All rectangles are parallelograms."

"All squares are both rhombi and rectangles."

We can also state other relationships like:

"All rhombi are not squares". (Notice that in the Venn diagram, there is a region (red area) that represents the rhombi outside of the region that represents the squares. This region represents all rhombi that are not squares.)

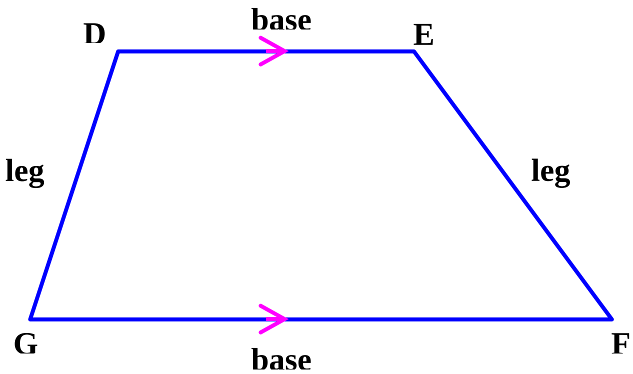
"All parallelograms are not rhombi". (Notice that in the Venn diagram, there is a region that represents the parallelograms (the light purple and blue areas) that falls outside the region that represents the rhombi. This region represents all the parallelograms that are not rhombi.)

Example 3: True or False? "Every rectangle is a square."

Even though rectangles have four right angles, they do not always have four congruent sides. Thus, since rectangles are a square only some of the time, this statement is false!

Trapezoids

trapezoid – A trapezoid is a quadrilateral with exactly one pair of sides parallel.



The parallel sides are called the bases. The non-parallel sides are called the legs.

*Note: The arrows mark parallel sides.

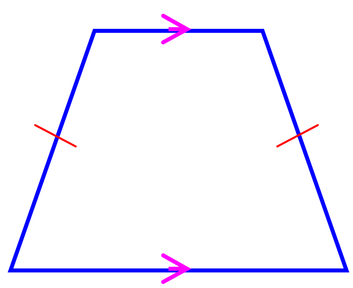
$\angle D$ and $\angle E$ are a pair of base angles.

$\angle G$ and $\angle F$ are a pair of base angles.

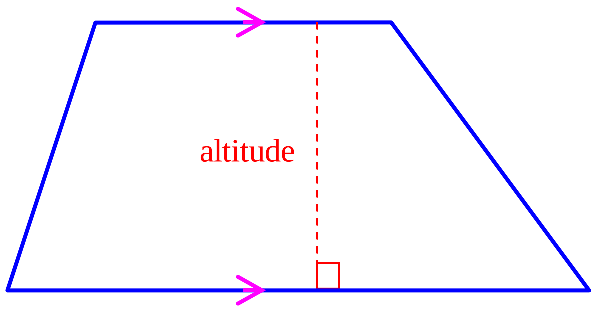
$\overline{DE}$ and $\overline{GF}$ are bases.

$\overline{DG}$ and $\overline{EF}$ are legs.

isosceles trapezoid – An isosceles trapezoid is a trapezoid with congruent legs.

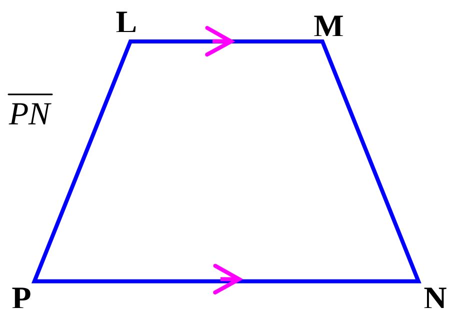


altitude of trapezoid – An altitude of a trapezoid is a perpendicular line drawn from any point on one of the parallel sides to the opposite side (or an extension of the opposite side).



Theorem 15-D

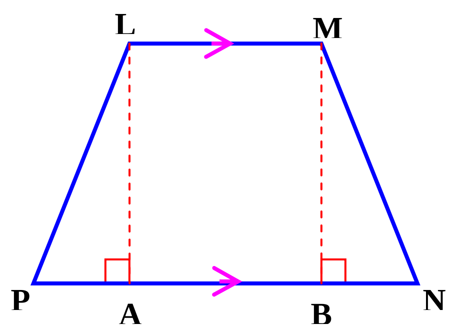
In an isosceles trapezoid, both pairs of base angles are congruent.



Given: Isosceles Trapezoid *LMNP* with bases $\overline{LM}$ and $\overline{PN}$

Prove: $\angle P \cong \angle N$ *and* $\angle PLM \cong \angle NML$

Draw two auxiliary line segments, altitudes *LA* and *MB*.



Plan for the proof: Prove that triangle LAP and MBN are congruent, and then apply CPCTC and the Angle Addition Postulate.

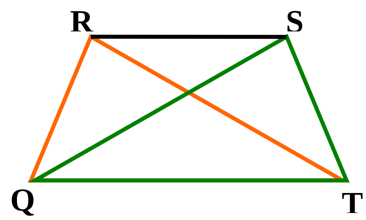
Statements	**Reasons**
1. Isosceles Trapezoid *LMNP*	Given
2. $\overline{LP} \cong \overline{MN}$ (Hypotenuse)	Definition of isosceles trapezoid
3. $\overline{LM} \parallel \overline{PN}$	Definition of trapezoid
4. $\overline{LA} \perp \overline{PN}$, $\overline{MB} \perp \overline{PN}$	Definition of altitude of trapezoid
5. $\overline{LA} \parallel \overline{MB}$	Theorem 8-D
6. $\overline{LA} \cong \overline{MB}$ (Leg)	Both segments represent the same length, the distance between two parallel lines; that is, a perpendicular line segment from a point on one line to the other line.
7. $\therefore \triangle LAP \cong \triangle MBN$	Hy-Leg Postulate (12-A)
8. Thus, $\angle P \cong \angle N$ (one pair of base angles)	CPCTC
9. $\angle PLA \cong \angle NMB$	CPCTC
10. $\angle ALM$ and $\angle BML$ are right angles.	Perpendicular lines form right angles.
11. $\angle ALM$ and $\angle BML$ are congruent.	Theorem 7-G
12. $\angle PLA + \angle ALM \cong \angle NMB + \angle BML$	Addition Property (Theorem 5-D)
13. $\therefore \angle PLM \cong \angle NML$ (other pair of base angles)	Angle Addition Postulate (1-B)

Theorem 15-E

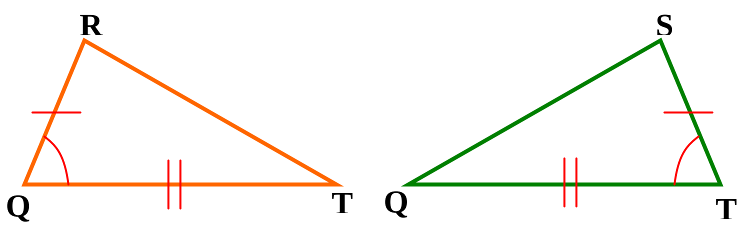
In an isosceles trapezoid, the diagonals are congruent.

Given: Isosceles Trapezoid *QRST*

Prove: $\overline{RT} \cong \overline{QS}$



We will separate the two overlapping triangles to help us visualize the two triangles more clearly; but, the references in the proof go back to the original isosceles trapezoid *QRST*.

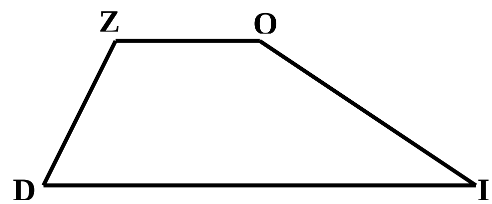


Statements	Reasons
Isosceles Trapezoid *QRST*	Given
$\overline{RQ} \cong \overline{ST}$	Definition of isosceles trapezoid
$\angle RQT \cong \angle STQ$	Theorem 15-D
$\overline{QT} \cong \overline{QT}$	Reflexive Property
$\triangle QRT \cong \triangle TSQ$	SAS
$\therefore \overline{RT} \cong \overline{SQ}$	CPCTC

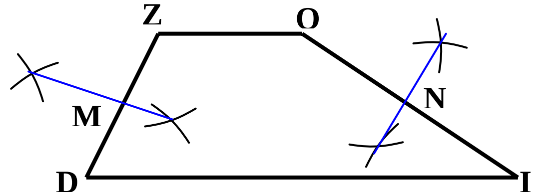
median – The median of a trapezoid is the line segment that joins the midpoints of the legs.

Construct the median of trapezoid ZOID using a straightedge and a compass.

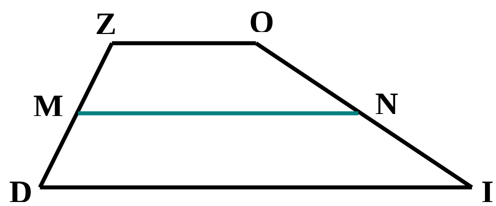
Step 1: Draw trapezoid ZOID.



Step 2: Draw a perpendicular bisector on $\overline{ZD}$ and $\overline{OI}$. Label the midpoints M and N.



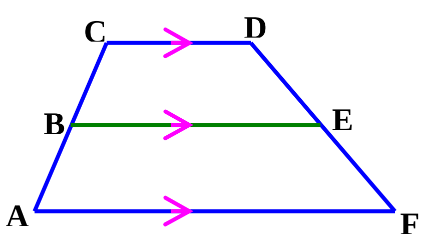
Step 3: Connect points M and N with $\overline{MN}$.



$\overline{MN}$ is the median of trapzoid *ZOID*.

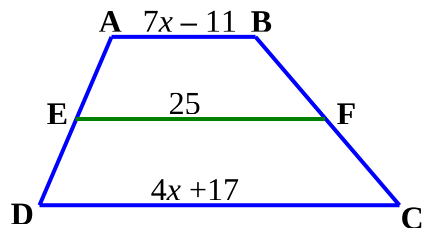
Theorem 15-F Mid-Segment Theorem

The median of a trapezoid is parallel to the bases and its length is one-half the sum of the lengths of the bases.



$$BE = \frac{1}{2}(CD + AF)$$

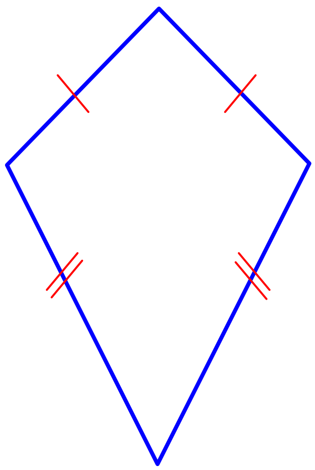
Example: For trapezoid ABCD shown below with median $\overline{EF}$, find the value of *x*.



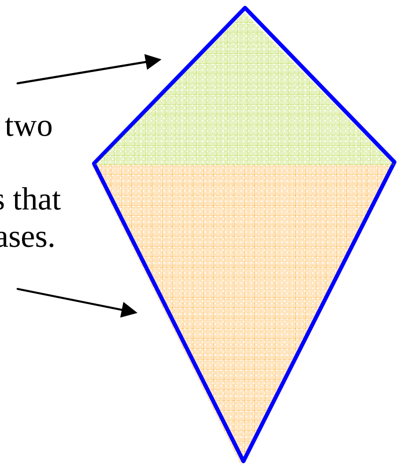
$EF = \frac{1}{2}(AB + DC)$	Theorem 15-F (Mid-Segment Theorem)
$25 = \frac{1}{2}\left[(7x - 11) + (4x + 17)\right]$	Substitution
$25 = \frac{1}{2}(7x - 11 + 4x + 17)$	Simplify
$25 = \frac{1}{2}(11x + 6)$	Collect like terms
$50 = 11x + 6$	Multiplication
$44 = 11x$	Subtraction
$4 = x$	Division

Kites

kite – A kite is a quadrilateral with exactly two distinct pairs of adjacent congruent sides.

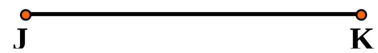


Think of a kite as two non-congruent isosceles triangles that connect at their bases.

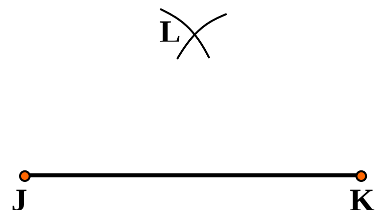


Let's take a look at how to construct a kite.

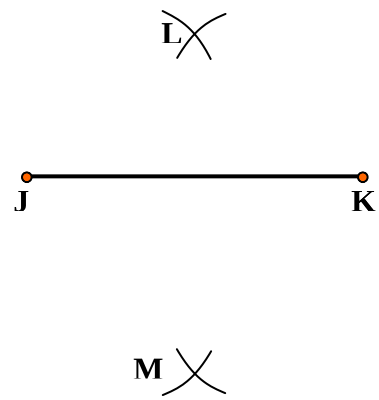
Step 1: With a straightedge, draw $\overline{JK}$.



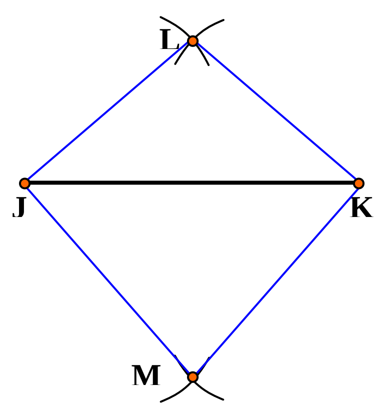
Step 2: Set the compass to a length more than half the length of $\overline{JK}$. Place the metal point of the compass at point J and draw an arc above $\overline{JK}$. Without changing the setting of the compass, move the metal point to point K and draw a second arc above $\overline{JK}$ that intersects with the first arc.



Step 3: Change the setting of the compass by increasing the distance between the metal point and the pencil point. Place the metal point of the compass at point J and draw an arc below $\overline{JK}$. Without changing the setting of the compass, move the metal point to point K and draw a second arc below $\overline{JK}$ that intersects with the other arc.

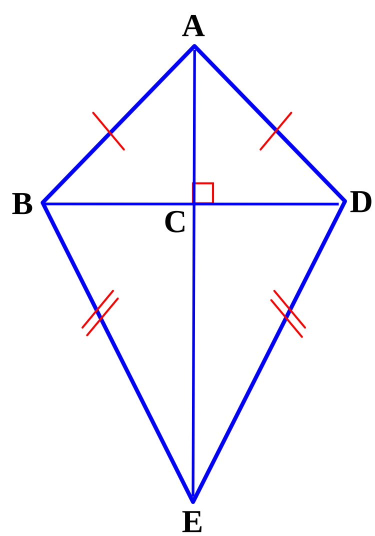


Step 4: Draw kite *JLKM*.



Theorem 15-G

The diagonals of a kite are perpendicular.



$\overline{AE} \perp \overline{BD}$

Graph Paper