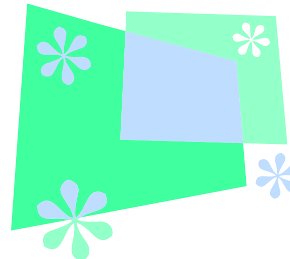
QUADRILATERALS, PARALLELOGRAMS, AND RECTANGLES

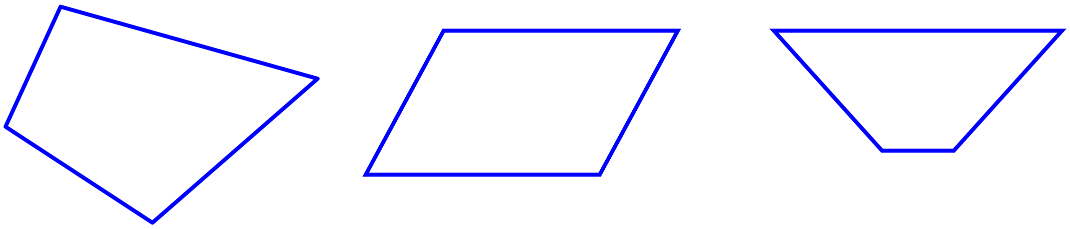
In this unit you will take an in-depth look at quadrilaterals, parallelograms, and rectangles. You will explore theorems and properties about parallelograms, special quadrilaterals that have parallel sides, and rectangles, special parallelograms with right angles.

Quadrilaterals

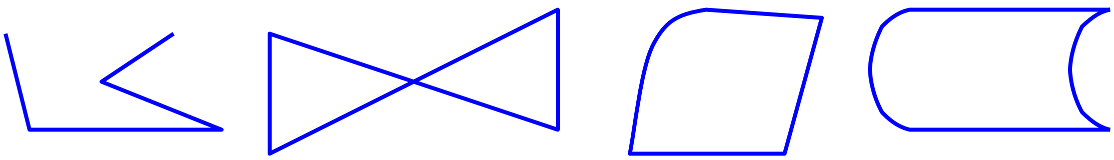


quadrilateral – A quadrilateral is a polygon with four sides.

Several examples of quadrilaterals are shown below.



Here are some figures that are NOT quadrilaterals.



A **B** **C** **D**

In **figure A**, the four-sided figure is not a polygon (not a closed figure).

In **figure B**, look closely to find four sides; but two sides overlap eliminating this figure as a single polygon.

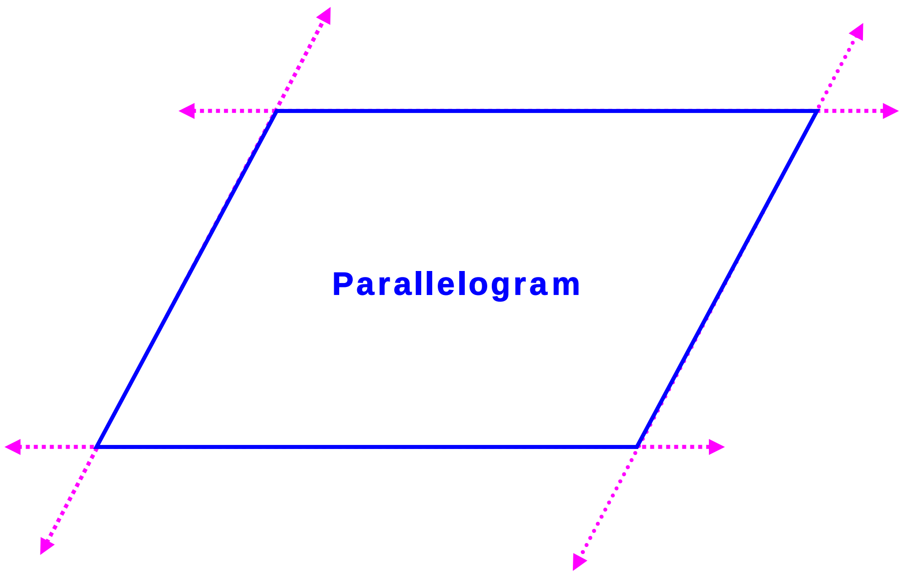
In **figure C**, there are only two segments, and the other side is one curved line.

In **figured D**, there are no distinct line segments.

Parallelograms

Some quadrilaterals are given other names because of the special angles and line segments that make up the shape.

parallelogram – A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

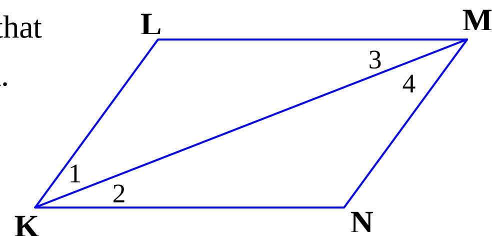


Theorem 14-A

The opposite sides of a parallelogram are congruent.

Given: $\square KLMN$ (This is a short way to write that quadrilateral $KLMN$ is a parallelogram.

Prove: $\overline{LM} \cong \overline{KN}$; $\overline{LK} \cong \overline{MN}$



Statements	**Reasons**
1. $\square KLMN$	Given
2. $\overline{LM} \parallel \overline{KN}$	Definition of parallelogram.
3. $\angle 2 \cong \angle 3$	Alternate interior angles of parallel lines are congruent.
4. $\overline{LK} \parallel \overline{MN}$	Definition of parallelogram.
5. $\angle 1 \cong \angle 4$	Alternate interior angles of parallel lines are congruent. (Rotate the parallelogram 90-degrees clockwise to visualize this better.)
6. $\overline{KM} \cong \overline{KM}$	Reflexive Property
7. $\triangle KLM \cong \triangle MNK$	ASA
8. $\overline{LM} \cong \overline{NK}$	CPCTC
9. $\overline{LK} \cong \overline{NM}$	CPCTC

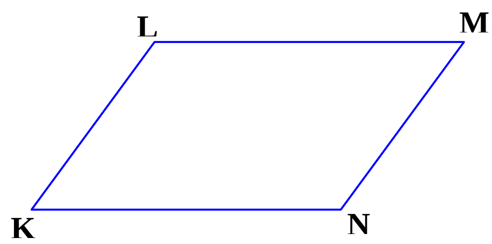
*Note: The theorem numbers are not written in the reasons. You can choose between writing out the entire theorem, an abbreviated version of the theorem, or just the theorem number.

Theorem 14-B

The opposite angles of a parallelogram are congruent.

Theorem 14-C

The consecutive pairs of angles of a parallelogram are supplementary.



Example 1: How does Theorem 7-L support Theorem 14-C?

Theorem 7-L states that consecutive interior angles of parallel lines are supplementary.

Angles *K* and *L* are consecutive angles for parallel lines *LM* and *KN*, thus they are supplementary.

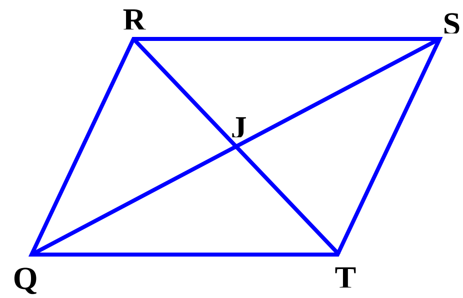
Angles *N* and *M* are consecutive angles for parallel lines *LM* and *KN*, thus they are supplementary.

Angles *L* and *M* are consecutive angles for parallel lines *KL* and *NM*, thus they are supplementary.

Angles *K* and *N* are consecutive angles for parallel lines *KL* and *NM*, thus they are supplementary.

Theorem 14-D

The diagonals of a parallelogram bisect each other.



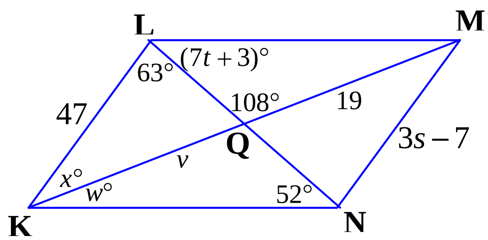
Given: $\square QRST$

Prove: $\overline{RJ} \cong \overline{JT}$; $\overline{QJ} \cong \overline{JS}$

Statements	**Reasons**
1. $\square QRST$	Given
2. $\overline{RS} \parallel \overline{QT}$	Definition of parallelogram.
3. $\angle RSQ \cong \angle SQT$	Alternate interior angles of parallel lines are congruent.
4. $\angle RJS \cong \angle QJT$	Vertical angles are congruent.
5. $\overline{RS} \cong \overline{QT}$	The opposite sides of parallelograms are congruent. (Theorem 14-A)
6. $\triangle RJS \cong \triangle QJT$	AAS
7. $\overline{RJ} \cong \overline{JT}$	CPCTC
8. $\overline{QJ} \cong \overline{JS}$	CPCTC

Theorem 14-E	**Either diagonal of a parallelogram separates the parallelogram into two congruent triangles.**

Example 2: Given ▱ *LMPN*. Solve for *s*, *t*, *v*, *w*, and *x*. Also determine the measure of angle *LMN*.



To find *s*, theorem 14-A states that the opposite sides of a parallelogram are congruent.

$47 = 3s - 7$ *$KL = NM$

$54 = 3s$

$18 = s$

$s = 18$

To find *t*, recall that the alternate interior angles of parallel lines are congruent.

$7t + 3 = 52$ *$m\angle MLN = m\angle LNK$

$7t = 49$

$t = 7$

To find *v*, theorem 14-D states that the diagonals of a parallelogram bisect each other.

$v = QM$ *$KQ = QM$

$v = 19$

To find *w*, first recall that vertical angles are congruent.

$m\angle LQM = 108$ *$m\angle LQM = m\angle KQN$

$m\angle KQN = 108$

Then, recall the Triangle Sum Theorem; that is, the sum of the angles in a triangle equals 180.

$108 + 52 + w = 180$ *$m\angle KQN + m\angle QNK + m\angle NKQ = 180$

$160 + w = 180$

$w = 20$

To find *x*, recall the Exterior Angle Sum Theorem; that is, the exterior angle of a triangle equals the sum of the two remote interior angles.

$108 = x + 63$ *$m\angle LQM \cong x + m\angle KLQ$

$45 = x$

$x = 45$

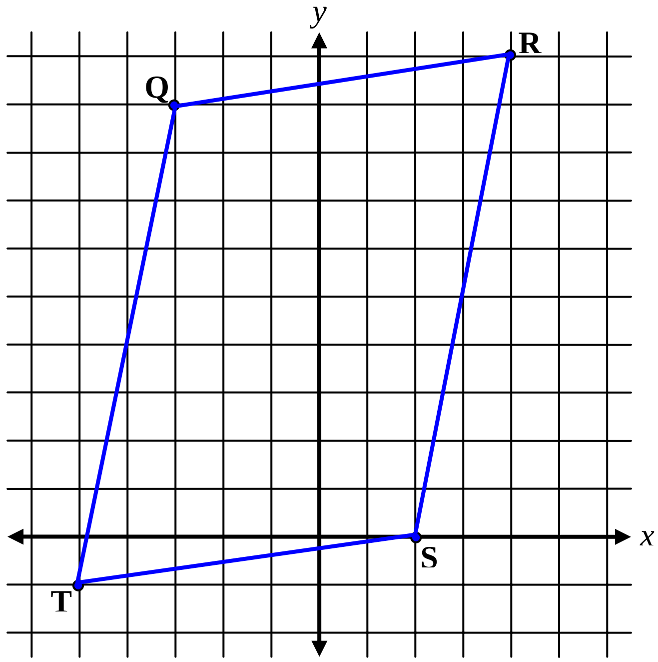
To find $m\angle LMN$, first determine the $m\angle LKN$ by recalling the Angle Addition Postulate, and then apply theorem 14-B; that is, opposite angles of a parallelogram are congruent.

$m\angle LKN = 45 + 20$ *$m\angle LKN = x + w$ (Angle Addition Postulate)

$m\angle LKN = 65$

$\therefore m\angle LMN = 65$ *$m\angle LKN = m\angle LMN$ (Theorem 14-B)

Example 3: Given quadrilateral *QRST* with vertices Q(–3,9), R(4,10), S(2,0), and T(–5, –1). Determine if Quadrilateral QRST is a parallelogram.



Recall postulate 8-A; that is, parallel lines have the same slope. To solve, determine and compare the slopes of the opposite segments in the quadrilateral. If the slopes are the same, then the segments are parallel.

First, let's compare the slopes of segments TS and QR.

Slope of $\overline{TS} = \dfrac{-1-0}{-5-2} = \dfrac{-1}{-7} = \dfrac{1}{7}$ Slope of $\overline{QR} = \dfrac{9-10}{-3-4} = \dfrac{-1}{-7} = \dfrac{1}{7}$

Since the slopes are the same, $\overline{TS} \parallel \overline{QR}$

Now, let's compare the slopes of segments TQ and SR.

Slope of $\overline{TQ} = \dfrac{-1-9}{-5-(-3)} = \dfrac{-10}{-2} = 5$ Slope of $\overline{SR} = \dfrac{0-10}{2-4} = \dfrac{-10}{-2} = 5$

Since the slopes are the same, $\overline{TQ} \parallel \overline{SR}$

Therefore, since both pairs of opposite sides are parallel, quadrilateral *QRST* is a parallelogram.

More Properties of Parallelograms

Theorem 14-F	**In a quadrilateral if both pairs of opposite sides are congruent, then the quadrilateral is a parallelogram.**

We can plan an approach for the proof of this theorem by looking at the end results and working backwards.

1) We want to end with a parallelogram.

2) We need to show that the opposite sides are parallel.

3) If we add a diagonal, an auxiliary line, we can use the theorems associated with parallel lines and transversals

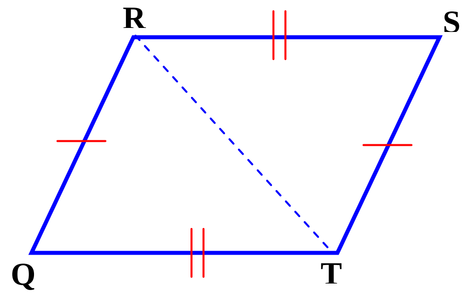
4) In particular, we need to recall theorem 8-A: “If the alternate interior angles of two lines cut by a transversal are congruent, then the lines are parallel.”

So, let’s get started!

Given: Quadrilateral QRST

$\overline{RS} \cong \overline{QT}$

$\overline{RQ} \cong \overline{ST}$



Prove: Quadrilateral QRST is a parallelogram.

Statements	**Reasons**
1. $\overline{RS} \cong \overline{QT}$; $\overline{RQ} \cong \overline{ST}$	Given
2. Draw auxiliary $\overline{RT}$	Two points determine a straight line.
3. $\overline{RT} \cong \overline{RT}$	Reflexive Property
4. $\triangle RQT \cong \triangle RST$	SSS
5. $\angle SRT \cong \angle RTQ$	CPCTC
6. $\therefore\ \overline{RS} \parallel \overline{QT}$	If the alternate interior angles of two lines cut by a transversal are congruent, then the lines are parallel. (Tm 8-A)*
7. $\angle QRT \cong \angle RTS$	CPCTC
8. $\therefore \overline{RP} \parallel \overline{ST}$	Theorem 8-A (This is seen more easily if you rotate the parallelogram 90-degrees clockwise.)
9. $\therefore$ Quadrilateral $QRST$ is a parallelogram.	Definition of parallelogram

*Tm is the abbreviation for "Theorem".

Theorem 14-G

In a quadrilateral if both pairs of opposite angles are congruent, then the quadrilateral is a parallelogram.

Theorem 14-H	**In a quadrilateral if its diagonals bisect each other, then the quadrilateral is a parallelogram.**

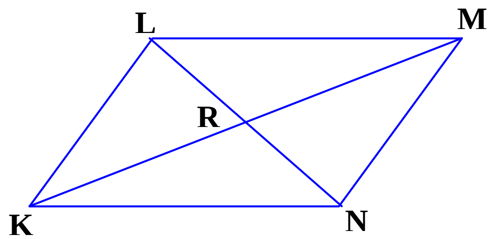
Example 4: Determine values for x and y, so that quadrilateral KLMN is a parallelogram.

$\overline{KR} = 4x - 3$

$\overline{RM} = 3y + 7$

$\overline{LR} = x$

$\overline{RN} = 2y - 15$



Apply Theorem 14 - H

$4x - 3 = 3y + 7$	LR must equal RN (Tm. 14-H)
$x = 2y - 15$	KR must equal RM (Tm. 14-H)

$4(2y - 15) - 3 = 3y + 7$	Substitute $2y - 15$ in for x in the first equation.
$8y - 60 - 3 = 3y + 7$	Distributive Property
$8y - 63 = 3y + 7$	Simplify
$5y - 63 = 7$	Subtraction Property
$5y = 70$	Addition Property
$y = 14$	Division Property
$x = 2(14) - 15$	Substitute
$x = 13$	Simplify

Check

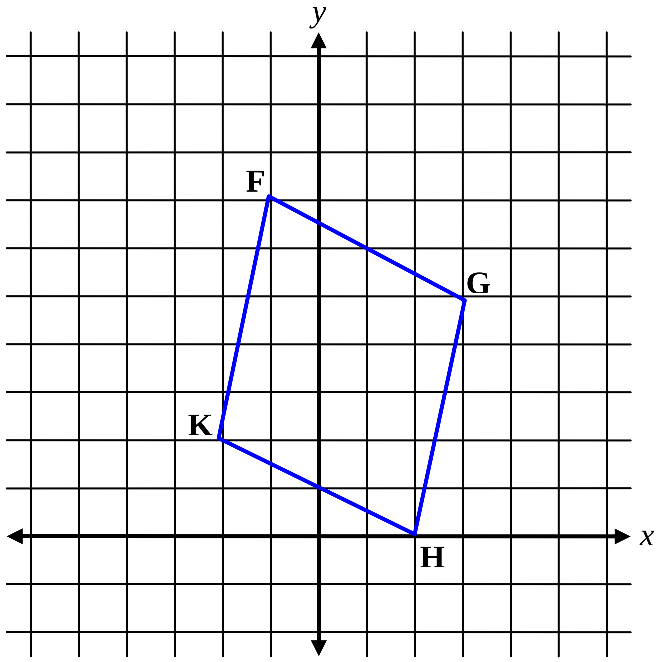
$LR = RN$	$KR = RM$
$4x - 3 = 3y + 7$	$x = 2y - 15$
$4(13) - 3 = 3(14) + 7$	$13 = 2(14) - 15$
$49 = 49$	$13 = 13$

When $x = 13$ and $y = 14$, Quadrilateral $KLMN$ is a parallelogram.

Theorem 14-I

In a quadrilateral if one pair of opposite sides is both congruent and parallel, then the quadrilateral is a parallelogram.

Example 5: Apply Theorem 14-I to determine if quadrilateral *FGHK*, with vertices F(–1,7), G(3,5), H(2,0), and K(–2, 2), is a parallelogram.



First we'll check a pair of opposites sides to see if they are parallel; then, we'll check to see if they are equal in measures.

1) Check the slopes of $\overline{KH}$ and $\overline{FG}$.

$$m \text{ of } \overline{KH} = \frac{2-0}{-2-2} = \frac{2}{-4} = -\frac{1}{2}$$

$$m \text{ of } \overline{FG} = \frac{7-5}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$$

The slopes are the same; therefore, $\overline{KH} \parallel \overline{FG}$.

2) Next we will check to see if the same two sides are congruent.

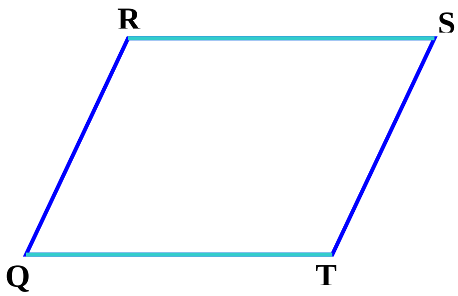
$$\begin{aligned} KH &= \sqrt{(2-0)^2 + (-2-2)]^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \end{aligned}$$

$$\begin{aligned} FG &= \sqrt{(7-5)^2 + (-1-3)^2} \\ &= \sqrt{4+16} \\ &= \sqrt{20} \end{aligned}$$

The segments are the same length, therefore they are congruent.

We have proven that $\overline{KH} \parallel \overline{FG}$ and $\overline{KH} \cong \overline{FG}$; therefore, by Theorem 14-I, quadrilateral *FGHK* is a parallelogram.

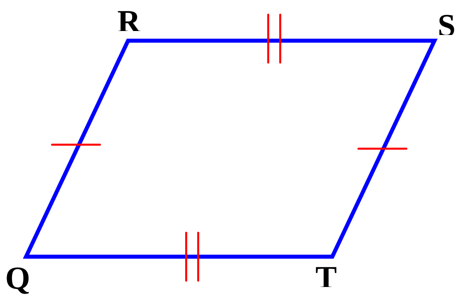
Summary of the Properties of a Parallelogram



Definition

$\overline{RS} \parallel \overline{QT}$

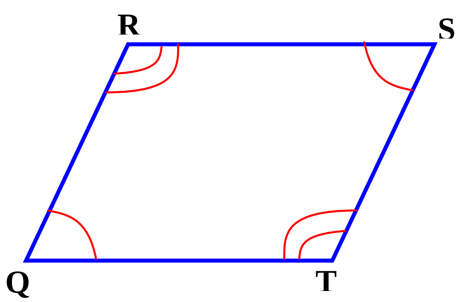
$\overline{RQ} \parallel \overline{ST}$



Theorem 14 - A

$RS = QT$

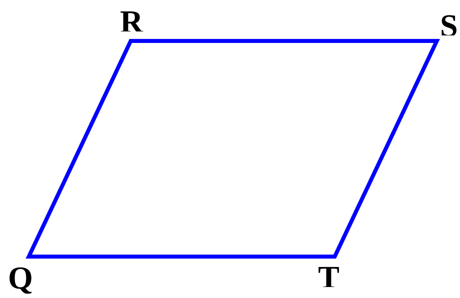
$RQ = ST$



Theorem 14 - B

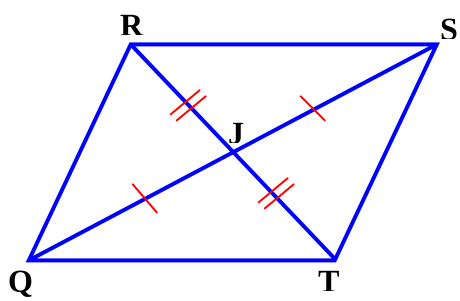
$m\angle R = m\angle T$

$m\angle Q = m\angle S$



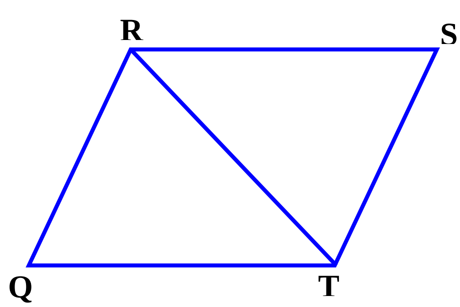
Theorem 14-C

$$m\angle R + m\angle S = 180$$
$$m\angle S + m\angle T = 180$$
$$m\angle Q + m\angle T = 180$$
$$m\angle R + m\angle Q = 180$$



Theorem 14-D

$$RJ = JT$$
$$QJ = JS$$



Theorem 14-E

$$\triangle QRT \cong \triangle RTS$$

In a parallelogram, what kind of triangles is created by either of the diagonals?
A diagonal divides a parallelogram into two *congruent* triangles. (**Theorem 14-F**)

Is a quadrilateral a parallelogram?
Yes, if both pairs of opposite sides are congruent. (**Theorem 14-G**)

Is a quadrilateral a parallelogram?
Yes, if its diagonals bisect each other. (**Theorem 14-H**)

Is a quadrilateral a parallelogram?
Yes, if one pair of sides are both parallel and congruent. (**Theorem 14-I**)

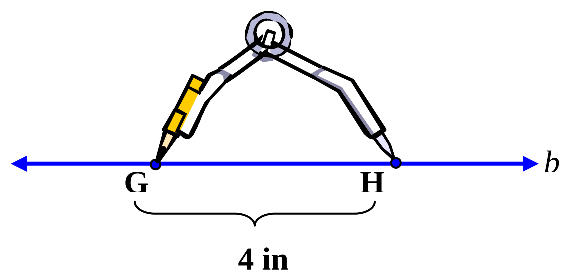
Rectangles

rectangle – A rectangle is a parallelogram with four right angles.

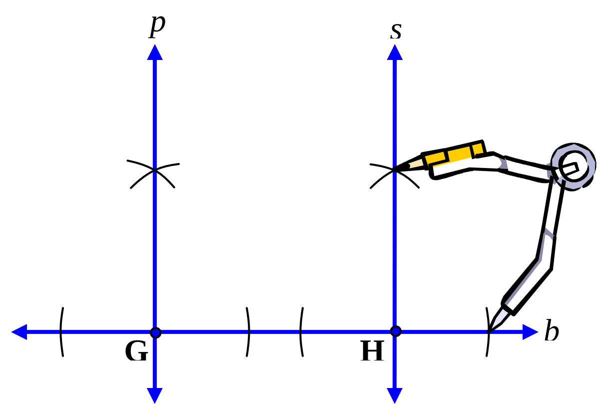
Theorem 14-J	**If a parallelogram is a rectangle, then its diagonals are congruent.**

We will construct a rectangle that is 4 inches long and 3 inches wide. We will then check to see if the diagonals are congruent.

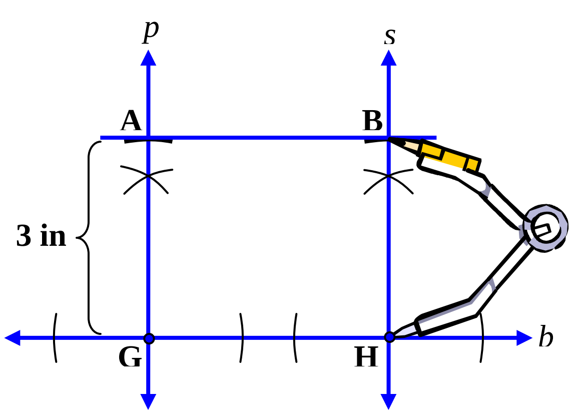
Step 1: Use a straightedge to draw line *b*. Label a point H on line b. On a ruler lay the metal point of the compass at 0 inches and then open your compass so that the pencil point touches the 4-inch mark on the ruler. Place the metal point of the compass at point H and mark point G so that GH measures 4 inches.



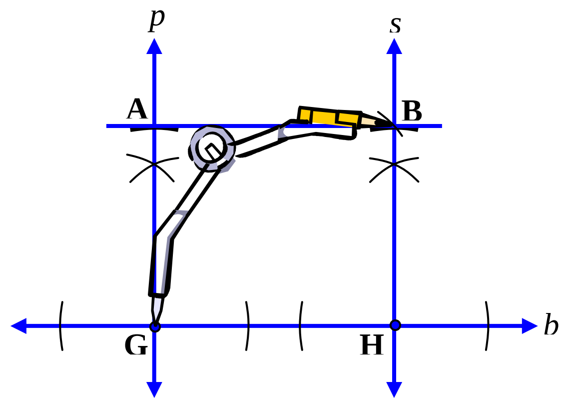
Step 2: Construct perpendicular lines to *b* at points G and H. Label the lines *p* and *s*.



Step 3: Using a ruler for reference, open the compass so that the distance between the metal point and the pencil point is 3 inches. Place the metal point of the compass at point G and mark off 3 inches on line *p*. Then move the metal point of the compass to point H and mark off 3 inches on line *s*. Draw $\overline{AB}$.



Step 4: Place the metal point of the compass at point G and stretch it so that the pencil point falls on point B. That is the length of one diagonal. Without changing the settings on the compass, move the metal point to point H and check to see if the pencil point falls on point A. The distance between points G and B should be the same as between points A and H.



Theorem 14-K

If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.