PROVING TRIANGLES CONGRUENT

In this unit you will examine how triangles can be proved that triangles are congruent. You will explore postulates and theorems related to triangles including the Side-Side-Side (SSS) postulate, Side-Angle-Side (SAS) postulate, the Angle-Side-Angle (ASA) postulate, and the Angle-Angle-Side (AAS) theorem. This unit will conclude with theorems about isosceles and equilateral triangles.

SSS Postulate

SAS Postulate

ASA Postulate

AAS Theorem

Isosceles and Equilateral Triangles

Side–Side–Side Postulate (SSS)

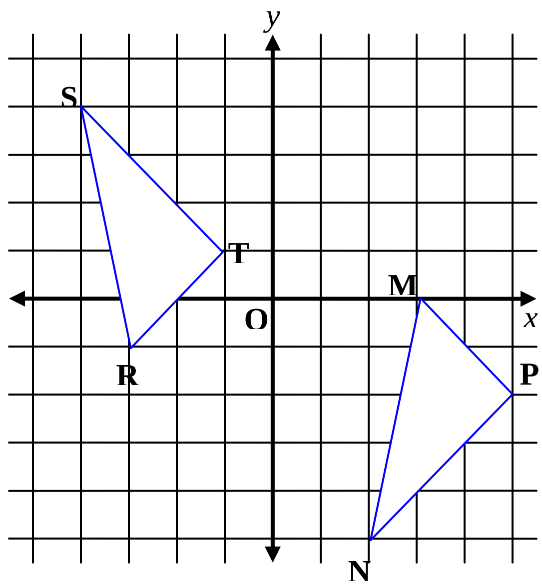
Postulate 11-A SSS Postulate	**If the sides of a triangle are congruent to the sides of a second triangle, then the triangles are congruent.**

Let's examine this postulate by looking at two triangles drawn in the coordinate plane. Determine if these triangles are congruent by calculating the length of the corresponding sides using the distance formula.

Given: $\triangle RST$ with vertices $R(-3,-1), S(-4,4),$ and $T(-1,1)$

$\triangle MNP$ with vertices $M(3,0), N(2,-5),$ and $P(5,-2)$

Determine if $\triangle RST \cong \triangle MNP$.



$\triangle RST$

$RS = \sqrt{[-4-(-3)]^2 + [4-(-1)]^2} = \sqrt{26}$

$ST = \sqrt{[-1-(-4)]^2 + (1-4)^2} = \sqrt{18}$

$TR = \sqrt{[-3-(-1)]^2 + (-1-1)^2} = \sqrt{8}$

$\triangle MNP$

$MN = \sqrt{(2-3)^2 + (-5-0)^2} = \sqrt{26}$

$NP = \sqrt{(5-2)^2 + [-2-(-5)]^2} = \sqrt{18}$

$PM = \sqrt{(3-5)^2 + [0-(-2)]^2} = \sqrt{8}$

$\overline{RS} \cong \overline{MN}$, $\overline{ST} \cong \overline{NP}$, and $\overline{TR} \cong \overline{PM}$

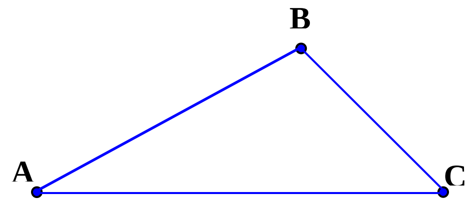
$\therefore \triangle RST \cong \triangle MNP$ by SSS postulate.

SSS	The three sides of one triangle must be congruent to the three sides of the other triangle.

You can construct a congruent triangle with a compass and a straight edge by applying the SSS postulate.

Given: $\triangle ABC$

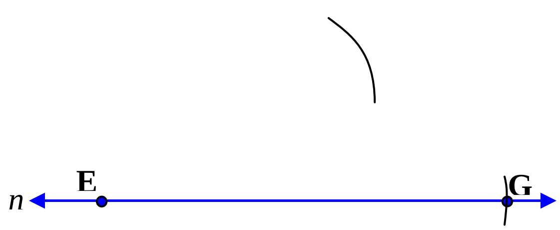
Construct: Congruent $\triangle EFG$



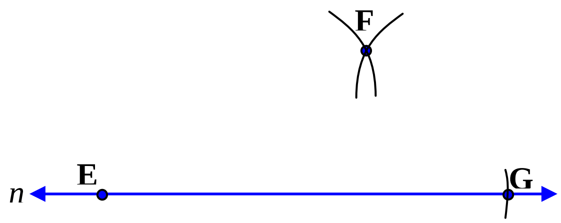
Step 1: Start with drawing a line and placing point E on the line.

Step 2: Move back to the triangle and place the metal point of the compass on point A and adjust the compass so that the pencil point touches point C. The compass will now be set to the length of segment AC. Without changing the setting of the compass, move back to the line. Place the metal point of the compass on point E and make an arc on the line. Name the point of intersection, point G.

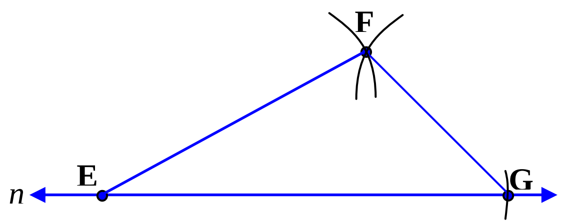
Step 3: Move the metal point of the compass back to point A of the triangle and adjust the compass so that the pencil point touches point B. The compass will now be set to the length of segment AB. Without changing the setting of the compass, move back to the line and place the metal point of the compass on point E. Make an arc above the line.



Step 4: Move the metal point of the compass to point C of the triangle and adjust the compass so that the pencil point touches point B. The compass will now be set to the length of segment BC. Without changing the setting of the compass, move back to the line and place the metal point of the compass on point G. Make an arc above the line that intersects the other arc. Name the point of intersection, point F.



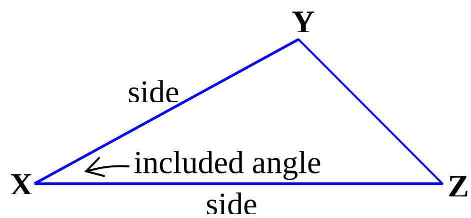
Step 5: Draw segments EF and GF.



$\triangle ABC \cong \triangle EFG$

Side–Angle–Side Postulate (SAS)

included angle – An included angle is the angle formed by two adjacent sides in a geometric figure.

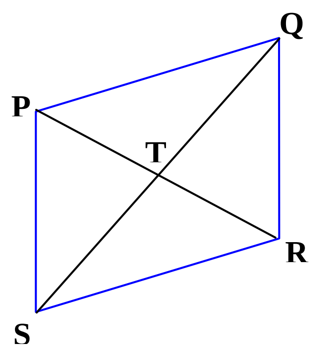


Postulate 11-B

SAS Postulate

If two sides and the included angle of a triangle are congruent to two sides and the included angle of a second triangle, then the triangles are congruent.

In the following proof, we will use the SAS Postulate to make the final conclusion of the proof. Thus, we will examine what is given and make statements supported by reasons that show the two corresponding sides of the triangles and the included angles are congruent.



Given: Point T is the midpoint of $\overline{PR}$.

Point T is the midpoint of $\overline{SQ}$.

Prove: $\triangle PTQ \cong \triangle RTS$

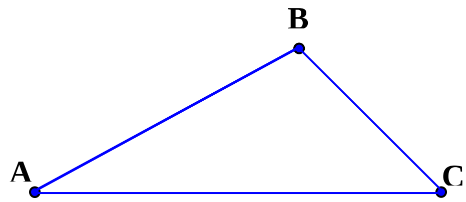
Statements	Reasons
1. Point T is midpoint of $\overline{PR}$.	Given
2. $\overline{PT} \cong \overline{TR}$	Definition of Midpoint
3. Point T is midpoint of $\overline{SQ}$.	Given
4. $\overline{ST} \cong \overline{TQ}$	Definition of Midpoint
5. $\angle PTQ \cong \angle STR$	Vertical angles are congruent. (Theorem 7-H)
6. $\triangle PTQ \cong \triangle RTS$	SAS

SAS	Two sides and the included angle of one triangle must be congruent to two sides and the included angle of the other triangle.

You can construct a congruent triangle with a compass and a straight edge by applying the SAS postulate.

Given: $\triangle ABC$

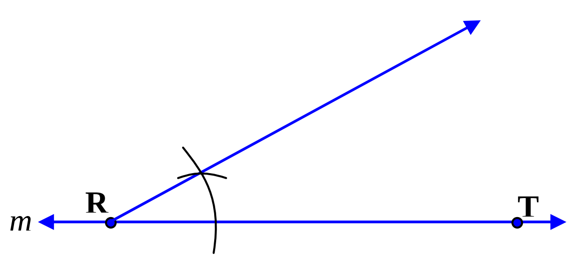
Construct: Congruent $\triangle RST$



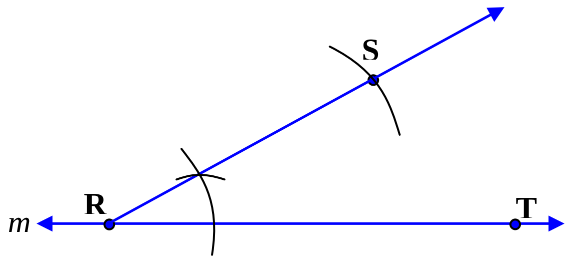
Step 1: Start with drawing a line *n* and placing point R on the line.

Step 2: Move back to the triangle and place the metal point of the compass on point A. Adjust the compass so that the pencil point touches point C. The compass will now be set to the length of segment AC. Without changing the setting on the compass, move back to the line and place the metal point of the compass on point R. Make an arc on the line and name the point of intersection, point T.

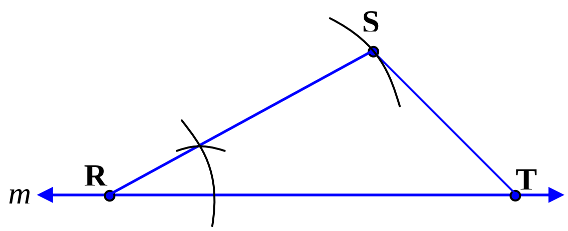
Step 3: Construct an angle at point R that is congruent to $\angle$A in the triangle. This angle is the **included** angle. (Refer back to a previous unit about “Angle Constructions”.)



Step 4: Move back to the triangle and place the metal point of the compass on point A. Adjust the compass so that the pencil point touches point B. The compass will now be set to the length of segment AB. Without changing the setting, move back to line *m* and place the metal point of the compass on point R. Make an arc on the top ray of the angle and name the point of intersection, point S.



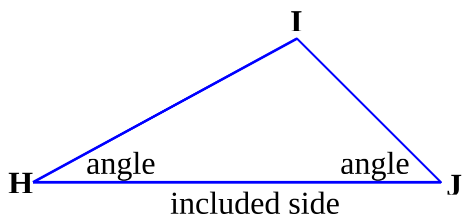
Step 5: In this step no further measuring is needed because all of the necessary constructions have been made that satisfy the SAS postulate. In the final step simply draw a segment to connect points S and T to complete the triangle.



$\triangle ABC \cong \triangle RST$

Angle–Side–Angle Postulate (ASA)

included side – An included side is the side that forms two different angles in a polygon.



Postulate 11-C

ASA Postulate

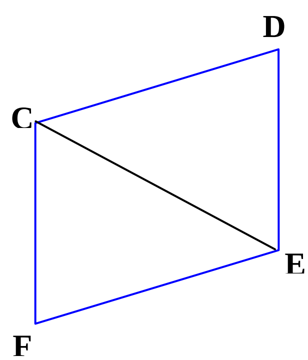
If two angles and the included side of a triangle are congruent to the two angles and included side of a second triangle, then the triangles are congruent.

In the following proof, we will use the ASA Postulate and CPCTC (Corresponding Parts of Congruent Triangles are Congruent) to make the final conclusions. Thus, we will examine what is given and make statements supported by reasons that show the two corresponding angles of the triangles and the included sides are congruent.

Given: $\overline{CD} \parallel \overline{FE}$

$\overline{CF} \parallel \overline{DE}$

Prove: $\angle F \cong \angle D$



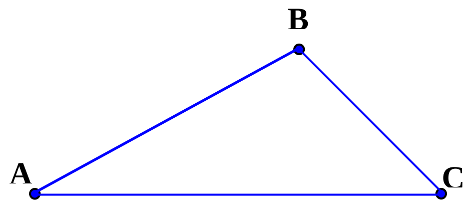
Statements	**Reasons**
1. $\overline{CD} \parallel \overline{FE}$	Given
2. $\angle DCE \cong \angle CEF$	Alternate interior angles of parallel lines (CD and FE) are congruent. (Theorem 7-J)
3. $\overline{CF} \parallel \overline{DE}$	Given
4. $\angle FCE \cong \angle CED$	Alternate interior angles of parallel lines (CF and DE) are congruent. (Theorem 7-J)
5. $\overline{CE} \cong \overline{CE}$	Reflexive Property
6. $\triangle FCE \cong \triangle DEC$	ASA (Postulate 11-C)
7. $\angle F \cong \angle D$	CPCTC

ASA	Two angles and the included side of one triangle must be congruent to two angles and the included side of the other triangle.

You can construct a congruent triangle with a compass and a straight edge by applying the ASA postulate.

Given: $\triangle ABC$

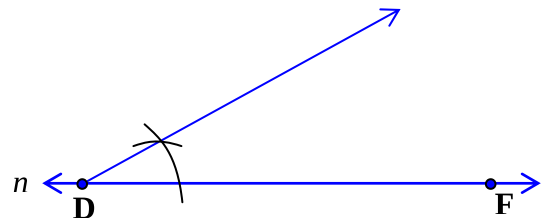
Construct: Congruent $\triangle DEF$



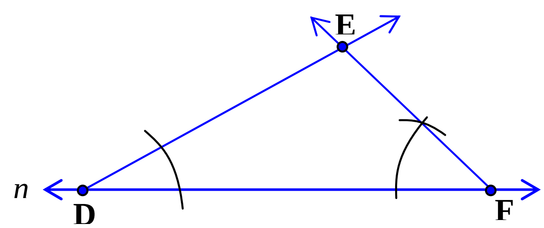
Step 1: Start with drawing a line n and placing point D on the line.

Step 2: Move back to the triangle and place the metal point of the compass on point A. Adjust the compass so that the pencil point touches point C. The compass will now be set to the length of segment AC. Moving back to the line, place the metal point of the compass on point D and make an arc on the line. Name the point of intersection, point F. Segment DF is the **included** side.

Step 3: Construct an angle at point D that is congruent to $\angle$A in the triangle. (Refer back to a previous unit about "Angle Constructions".)



Step 4: Construct an angle at point F that is congruent to $\angle$C in the triangle. (Refer back to a previous unit about "Angle Constructions".



Step 5: Name the point where the two rays of the constructed angles intersect, point E. The construction is complete.

$$\triangle ABC \cong \triangle DEF$$

Angle–Angle–Side Theorem (AAS)

Theorem 11-A AAS Theorem	If two angles and a non-included side of a triangle are congruent to two angles and a non-included side of a second triangle, then the two triangles are congruent.

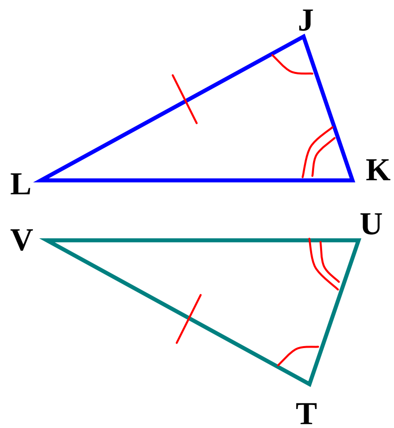
Proof of AAS Theorem

Given: $\angle J \cong \angle T$

$\angle K \cong \angle U$

$\overline{JL} \cong \overline{TV}$

Prove: $\triangle JKL \cong \triangle TUV$



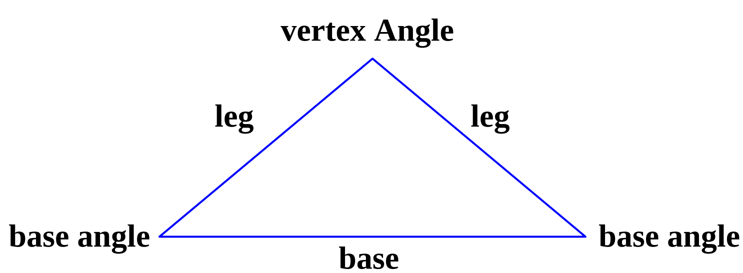
Statements	Reasons
1. $\angle J \cong \angle T$	Given
2. $\angle K \cong \angle U$	Given
3. $\overline{JL} \cong \overline{TV}$	Given
4. $\angle L \cong \angle V$	Third Angle Theorem (Theorem 10-B)
5. $\triangle JKL \cong \triangle TUV$	ASA (Postulate 11-C)

AAS	Two angles and a non-included side of one triangle must be congruent to the corresponding two angles and side of the other triangle.

Isosceles and Equilateral Triangles

Theorem 11-B **Isosceles Triangle Theorem**	**If two sides of a triangle are congruent, then the angles that are opposite those sides are congruent.**
Theorem 11-C	**If two angles of a triangle are congruent, then the sides that are opposite those angles are congruent.**
Corollary 11-B-1	**A triangle is equilateral if and only if it is equiangular.**
Corollary 11-B-2	**Each angle of an equilateral triangle measures 60°.**

Parts of an Isosceles Triangle



vertex angle – The vertex angle of an isosceles triangle is the angle formed by the two congruent sides.

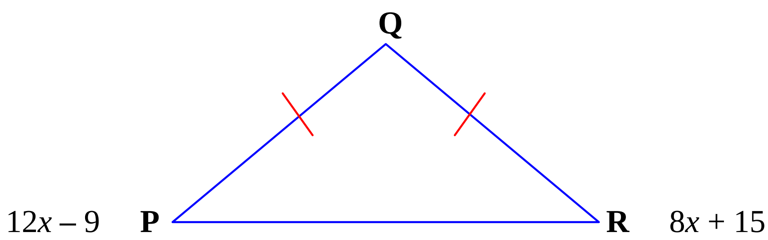
leg – The leg of an isosceles triangle is one of the two congruent sides.

base angle – The base angle of an isosceles triangle is an angle that is opposite one of the congruent sides.

base – The base of an isosceles triangle is the included side of the two congruent angles.

Example: Find the measure of each angle in isosceles triangle PQR. Segment PR is the base, $m\angle P = 12x - 9$, and $m\angle R = 8x + 15$.

Draw a picture and label the given parts. This will help to solve the problem.



$m\angle P = m\angle R$	Isosceles Triangle Theorem (Theorem 11-B)
$m\angle P = 12x - 9$	Given
$m\angle R = 8x + 15$	Given
$12x - 9 = 8x + 15$	Substitution Property
$12x = 8x + 24$	Addition Property (Add 9 to both sides.)
$4x = 24$	Subtraction Property (Subtract 8x from both sides.)
$x = 6$	Division Property (Divide both sides by 4.)

To find the measure of angles *P* and *R*, substitute *x* = 6 into the expressions that represent the size of the angles:

$m\angle P = 12x - 9$	$m\angle R = 8x + 15$
$m\angle P = 12(6) - 9$	$m\angle R = 8(6) + 15$
$m\angle P = 63°$	$m\angle R = 63°$

To find angle *Q*, use the Angle Sum Theorem:

$m\angle P + m\angle Q + m\angle R = 180$	Angle Sum Theorem (Theorem 10-A)
$63 + m\angle Q + 63 = 180$	Substitution
$m\angle Q + 126 = 180$	Simplify
$m\angle P = 54°$	Subtraction (Subtract 126 from both sides.)

Therefore, in isosceles triangle *PQR* the measure of each of the two base angles is 63 degrees and the measure of the vertex angle is 54 degrees.