COUNTING AND ARRANGEMENTS

This unit is about various counting techniques to calculate probability and the number of outcomes. The Fundamental Counting Principal is the underlying principle for determining the number of possible outcomes. There are two types of counting arrangements: permutations and combinations. A permutation is an arrangement of objects in which the order of the arrangement is important to the number of outcomes. A combination is an arrangement of objects where order is not taken into account and results in fewer outcomes than permutations.

Fundamental Counting Principal

Factorial Numbers

Permutations

Combinations

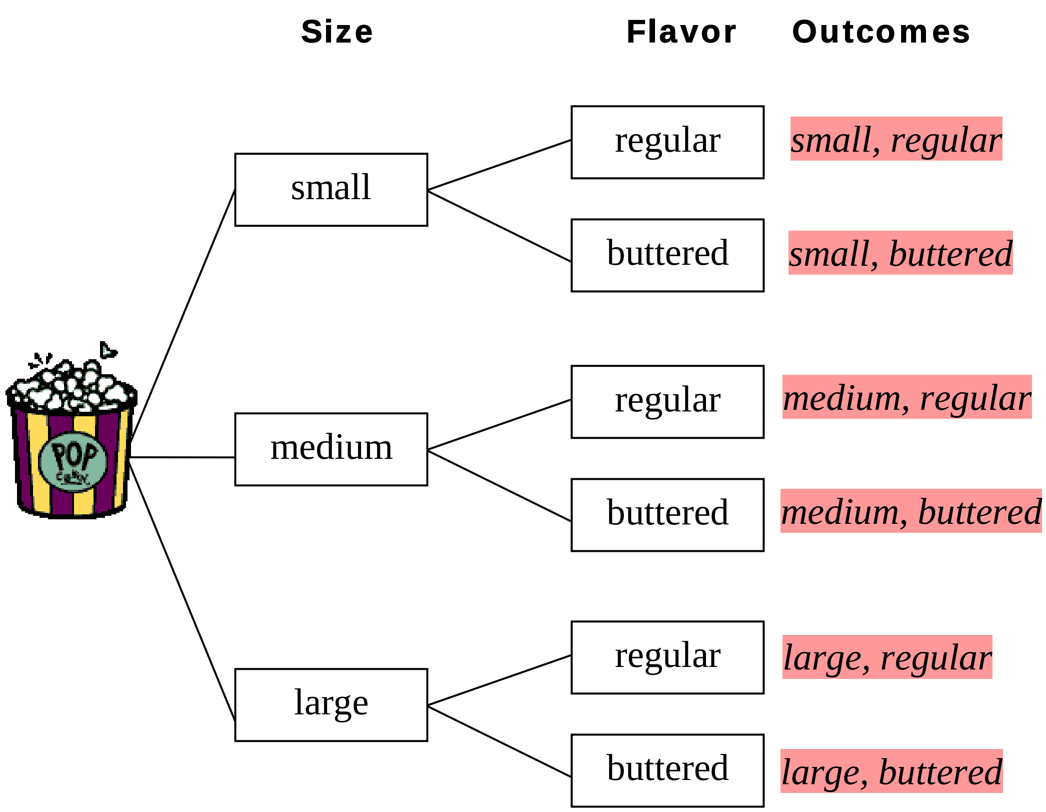
Fundamental Counting Principle

> **Fundamental Counting Principle**
>
> If there are *m* ways that one event can occur and *n* ways that another event can occur, then there are ***m x n*** ways that both events can occur.

Example 1: A movie theater sells popcorn in small, medium, or large containers. Each size is also available in regular or buttered popcorn. How many options for buying popcorn does the movie theatre provide?

First consider the three options for size: small, medium, and large. Then consider that each size of popcorn can be the regular flavor or the buttered flavor.

In the figure below, the diagram first shows the three sizes and then the two possible flavors for each size. The resulting combinations are listed under the "outcomes". Study the diagram to determine the total number of outcomes.



There are ***six*** possible options for buying popcorn at the movie theatre. They are small and regular, small and buttered, medium and regular, medium and buttered, large and regular, and large and buttered.

The Fundamental Theorem can be used to determine the number of outcomes quickly.

$$\text{size} \times \text{flavor} = \text{number of outcomes}$$

$$3 \times 2 = 6$$

Example 2: If the theater in the previous problem adds three new flavors, caramel apple, jelly bean, and bacon cheddar, to the popcorn choices, how many options will the customers have?

Apply the Fundamental Theorem to determine the number of outcomes.

$$\text{size} \times \text{flavor} = \text{number of outcomes}$$

$$3 \times 5 = 15$$

There are a total of 15 options.

Example 3: Emily is choosing a password for access to her Internet account. She decides to use a combination of digits and letters; but she doesn't want to use the digit 0 or the letters A, E, I, O, or U. Each letter or number may be used more than once. The password must be 3 letters followed by 2 digits. How many choices does Emily have?

Consider there are 26 letters in the alphabet, but Emily is choosing not to use five of them (A, E, I O, and U); thus, there are 21 possible letters from which to choose.

Also consider there are 10 single digit numbers (0 through 9), but Emily is choosing not to use one of them (0); thus, there are 9 possible digits from which to choose.

Apply the fundamental counting principle to determine the number of choices Emily will have in determining the password. There are 21 possible letters and 9 possible digits.

first letter		*second letter*		*third letter*		*first digit*		*second digit*
?	×	?	×	?	×	?	×	?
21 choices		21 choices		21 choices		9 choices		9 choices

$$21 \times 21 \times 21 \times 9 \times 9 = 750{,}141$$

There are 750,141 passwords that Emily could choose!

Factorial Numbers

When considering the arrangement of objects we use permutations or combinations. Before we look at permutations and combinations, we need to understand factorial numbers. Let's take a look.

$n!$ is read "n factorial"

For example, $6!$ means $6 \times 5 \times 4 \times 3 \times 2 \times 1$.

This is the product of all the whole numbers starting with 6 and going down to 1.

$6 \times 5 = 30$

$30 \times 4 = 120$

$120 \times 3 = 360$

$360 \times 2 = 720$

$720 \times 1 = 720$

$6! = 720$

Example: Evaluate $10!$

$10!$ means $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$10! = 3,628,800$

Permutations

A **permutation** is an arrangement of objects in a specific order. When objects are arranged in a row, the permutation is called a **linear permutation**.

First, let's consider an arrangement of a given number of items and arranging ALL of them in as many ways as possible.

> **Permutations of *n* Objects**
>
> The number of permutations of n objects is given by $n!$
> (! Is called factorial and means to multiply all consecutive integers starting with n.)

Example 1: On a baseball team, nine players are designated as the starting line up. Before a game, the coach announces the order in which the nine players will bat. How many different batting orders are possible?

Think: First there are 9 player choices, and then once a player is chosen, there are then 8 players left from which to choose. Once that player is chosen, there are 7 choices left, and so on.

Mathematically, this looks like: $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Using factorial notation:

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$= 362,880$ possible batting orders.

When choosing from nine players, there are 362,880 possible batting orders.

*To use a scientific calculator, find the factorial key (!) or (n!). Just press the number 9, and then the factorial key.

Now, let's consider an arrangement of a given number of items and arranging a specified number of these items in as many ways as possible.

Permutations of *n* Objects Taken *r* at a Time

The number of permutations of n objects taken r at a time, denoted by $P(n, r)$ or ${}_nP_r$ is given by:

$$P(n,\ r) = {}_nP_r = \frac{n!}{(n-r)!},\ \text{where } r \leq n$$

Example 2: There are 8 finalists in a spelling bee. Each student is given a number to determine the order in which he or she will compete. How many ways can the spelling order of the first, second, and third positions be assigned?

In this problem, order is important. The students will be arranged in a specific order. We are determining how many arrangements of 8 students can be made in the first, second, and third positions. Thus, this problem is a permutation of 8 items taken 3 at a time.

$${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_8P_3 = \frac{8!}{(8-3)!}$$

$$= \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = \frac{8 \cdot 7 \cdot 6}{1}$$

$$= 336$$

There are 336 ways the eight students could be assigned first, second, and third places.

Now, let's revisit arranging a given number of items and arranging ALL of them in as many ways as possible. This time we'll use the formula.

Example 3: How many ways can the letters of the word "random" be arranged?

R A N D O M

Order is important in arranging the letters and we are arranging ALL of them. Thus, this problem is a permutation of 6 items taken 6 at a time.

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} \\
{}_6P_6 &= \frac{6!}{(6-6)!} \\
&= \frac{6!}{0!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} \\
&= 720
\end{aligned}
$$

*0! is accepted as equal to 1.

*Sometimes, the formula for this special case of *n* items take *n* at a time is written as follows:

$$
\begin{aligned}
{}_nP_n &= n! \\
{}_6P_6 &= 6! = 720
\end{aligned}
$$

There are 720 ways the letters in the word "random" may be arranged.

Example 4: If we look at arranging letters in the word "success", we need to realize that when an s or c is selected, it does not matter which is which. So there are less ways to select the arrangement. How many ways can the letters of the word "success" be arranged?

S U C C E S S

This problem is an example of a permutation with repetition. The formula for this problem is:

$P = \frac{n!}{a!b!}$ where "*a*" and "*b*" are repeating letters.

We use all 7 letters but the "s" is repeated 3 times and the "c" is repeated twice.

$n = 7 \qquad a = 3 \qquad b = 2$

$$P = \frac{7!}{3!2!}$$

$$P = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1}$$

$$P = \frac{7 \times 6 \times 5 \times \cancel{4}^{2} \times \cancel{3} \times \cancel{2} \times 1}{\cancel{3} \times \cancel{2} \times 1 \times \cancel{2} \times 1}$$

$$P = \frac{7 \times 6 \times 5 \times 2}{1}$$

$$P = 420$$

The letters in the word "success" may be arranged 420 different ways.

Combinations

An arrangement of objects in which order is **NOT** important is called a combination.

Combinations of *n* Objects Taken *r* at a Time

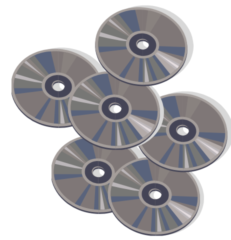
The number of combinations of n objects taken r at a time is given by:

$$C(n,r) = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ where } 0 \le r \le n$$

$C(n,r)$, ${}_nC_r$ and $\binom{n}{r}$ have the same meaning.

All are read “*n* choose *r*”.

Example: Find the number of ways to choose 6 different CD’s from a selection of 18 CD’s.



We have 18 CD’s that can be chosen in any order. So, order does NOT matter in this problem; thus, the arrangement in this problem is a combination.

$n = 18 \qquad r = 6$

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

$${}_{18}C_6 = \frac{18!}{6!(18-6)!}$$

$$= \frac{18!}{6!12!}$$

$$= \frac{18\cdot17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$$

$$= \frac{\cancel{18}^{3}\cdot17\cdot\cancel{16}^{4}\cdot\cancel{15}^{\cancel{5}^{1}}\cdot\cancel{14}^{7}\cdot13\cdot\cancel{12}\cdot\cancel{11}\cdot\cancel{10}\cdot\cancel{9}\cdot\cancel{8}\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{12}\cdot\cancel{11}\cdot\cancel{10}\cdot\cancel{9}\cdot\cancel{8}\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1}$$

$$= 3\cdot17\cdot4\cdot7\cdot13$$

$$= 18{,}564$$

There are 18,564 different ways that 6 CD’s can be chose from a selection of 18 CD’s.

The solution to this problem can be written in a simpler way.

$$_{18}C_6 = \frac{18!}{6!(18-6)!}$$

$$= \frac{18!}{6!12!}$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 12!}$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot \cancel{12!}}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{12!}}$$

$$= \frac{\cancel{18}^{3} \cdot 17 \cdot \cancel{16}^{4} \cdot \cancel{15}^{\cancel{5}^{1}} \cdot \cancel{14}^{7} \cdot 13 \cdot \cancel{12!}}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{12!}}$$

$$= 3 \cdot 17 \cdot 4 \cdot 7 \cdot 13$$

$$= 18{,}564$$