TRANSFORMATIONS

When transforming shapes in the *x*-*y* coordinate plane, shapes may be translated, reflected, rotated, and/or dilated. Some shapes may change in orientation while others may change in size by increasing or decreasing. There is a before and an after view to compare and observe the various transformations.

Types of Transformations

Translations

Reflections and Symmetry

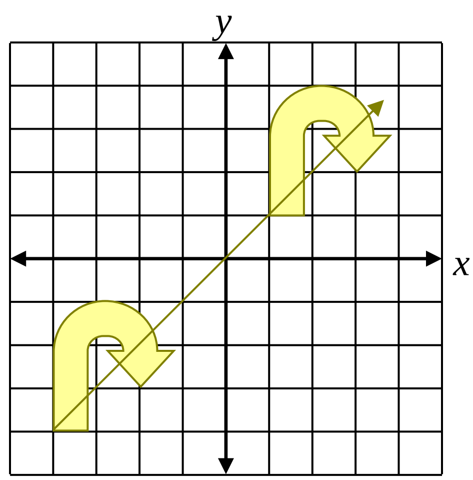
Rotations

Dilations

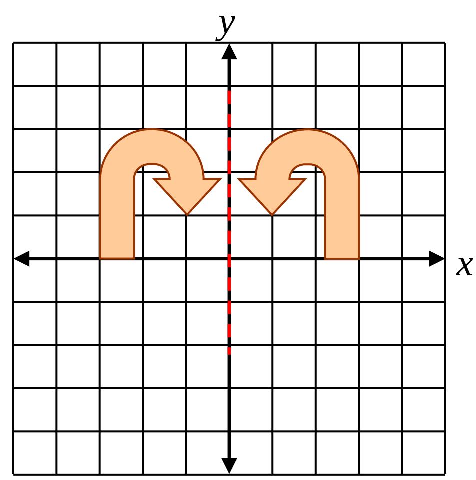
Types of Transformations

A ***transformation*** is a change made to the size or location of a figure. The new figure formed by the translation is called an ***image***.

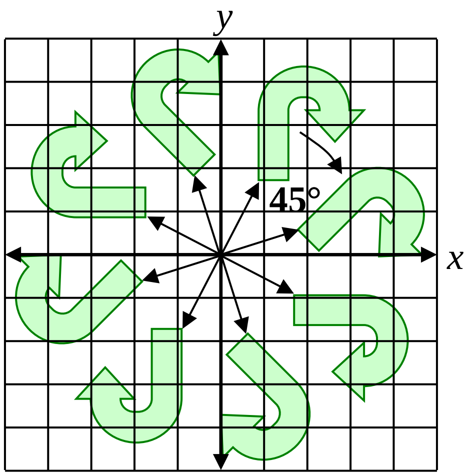
Four types of transformations are *translations, reflections, rotations, and dilations.*



Translation - sliding a figure along a straight line without turning to another location.



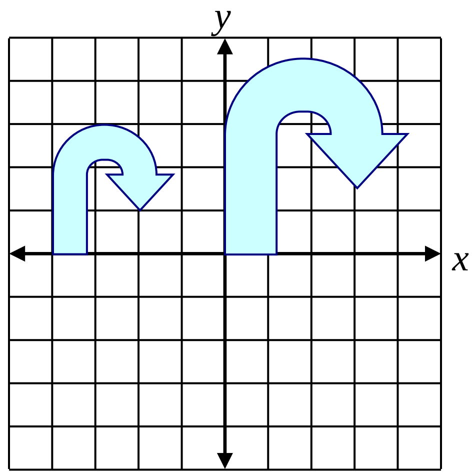
Reflection - flipping a figure over a line of reference creating a mirror image.



Rotation - turning a figure around a fixed point called the center of rotation.

The **angle of rotation** is an angle formed by rays drawn from the center of rotation through corresponding points on an original figure and its image.

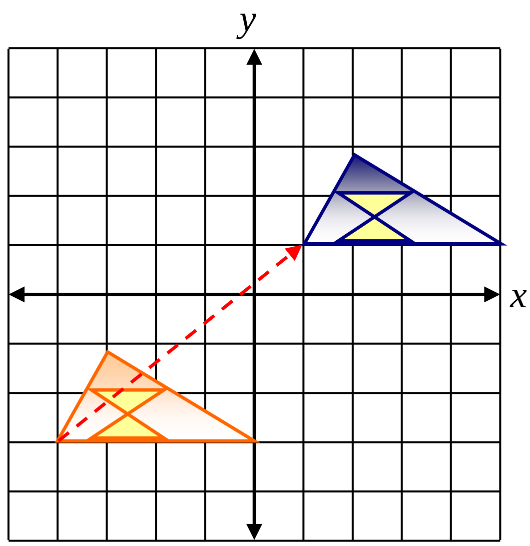
This curved arrow has been rotated 45° each time around the center of rotation, $(0,0)$.



Dilation - enlarging or reducing a figure.

Translations

A **translation** is a transformation in which each point of the figure moves the same distance in the same direction. A figure and its image are congruent.



A translation slides a figure along a line without turning.

Example 1: Describe the translation in words.

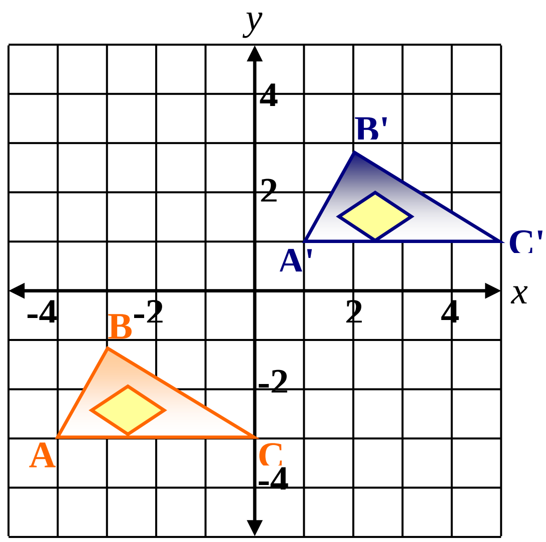


Figure ABC is translated to figure A'B'C'. (A'B'C' is read "A-primed, B-primed, C-primed".) The translation is 5 units to the right and 4 units up.

To check, start at point **A** and count **five** units to the **right**, and then count **four** units **up**. You will end up at point **A'**. Do the same for the other two pairs of corresponding points.

Notation in the Coordinate Plane: The translation of each point (x, y) of a shape can be represented using coordinate notation.

Translation: $(x, y) \rightarrow (x + a, y + b)$

$a =$ how many units a point moves horizontally

$b =$ how many units a point moves vertically

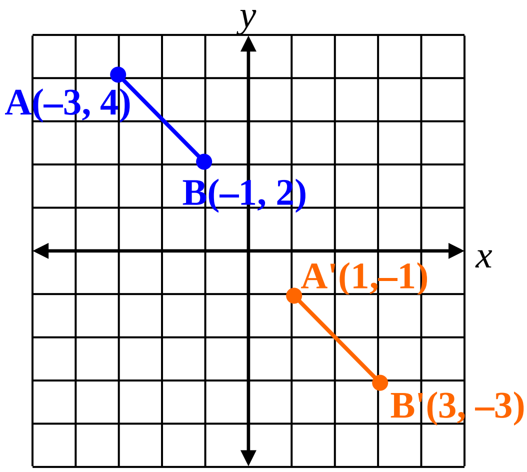
If $a > 0$ (positive), the point moves to the right.

If $a < 0$ (negative), the point moves to the left.

If $b > 0$ (positive), the point moves up.

If $b < 0$ (negative), the point moves down.

Example 2: Given segment AB with vertices A(–3, 4) and B(–1, 2), find the coordinates of the image after the translation $(x, y) \rightarrow (x+4, y-5)$ and draw the image.



Translation:

$(x, y) \rightarrow (x+a, y+b)$

$(x, y) \rightarrow (x+4, y-5)$

$A(-3,4) \rightarrow A'(-3+4,\ 4-5) = A'(1,-1)$

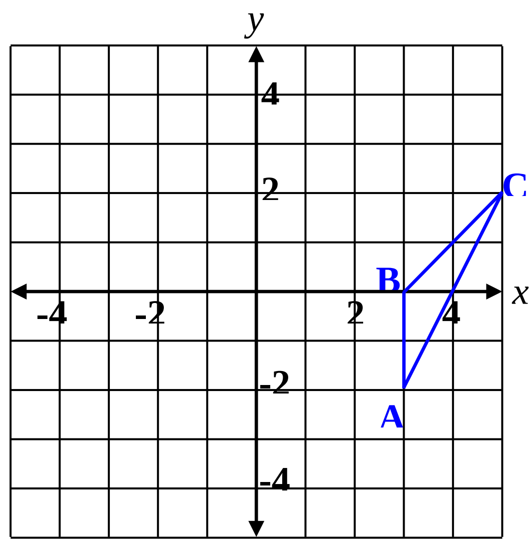
$B(-1,2) \rightarrow B'(-1+4,\ 2-5) = B'(3,-3)$

The endpoints of the translated segment are $A'(1,-1)$ and $B'(3,-3)$.

Example 3: Draw triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2). Then find the coordinates of the vertices of the image after the given translation. Draw the image.

$\text{Translation}: \ (x, y) \rightarrow (x - 7, y + 2)$

Step 1: Graph triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2).



Step 2: Find the coordinates of the vertices of the image.

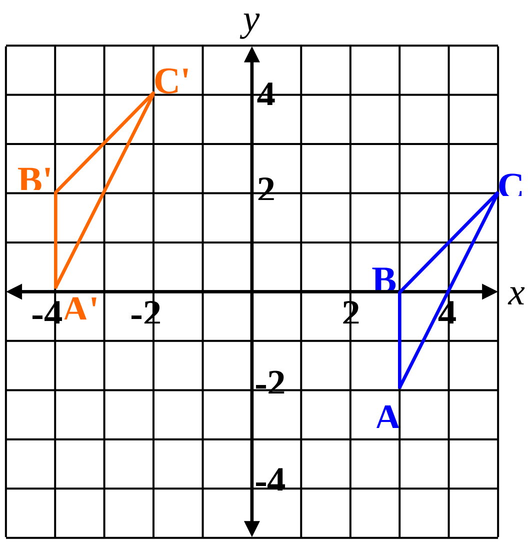
$\text{Translation}: \ (x, y) \rightarrow (x - 7, y + 2)$

Subtract 7 from each *x*-coordinate. Add 2 to each *y*-coordinate.

Original		Image
$P(x, y)$	$\rightarrow$	$P'(x-7, y+2)$
$A(3, -2)$	$\rightarrow$	$A'(-4, 0)$
$B(3, 0)$	$\rightarrow$	$B'(-4, 2)$
$C(5, 2)$	$\rightarrow$	$C'(-2, 4)$

Step 3: Draw the image, A'B'C'.

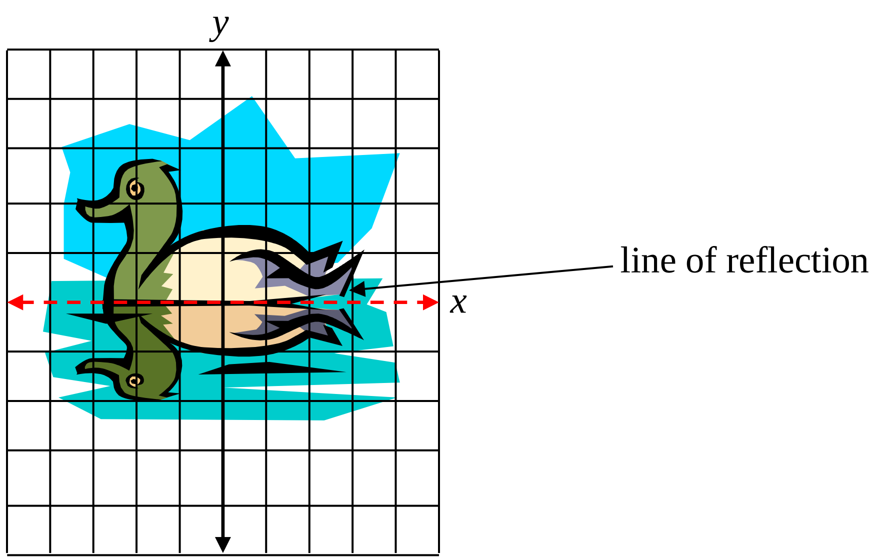
Each point on triangle ABC moves 7 units to the left and 2 units up to translate to triangle $A'B'C'$.



The vertices of the translated triangle are $A'(-4, 0), B'(-4, 2)$, and $C'(-2, 4)$.

Reflections and Symmetry

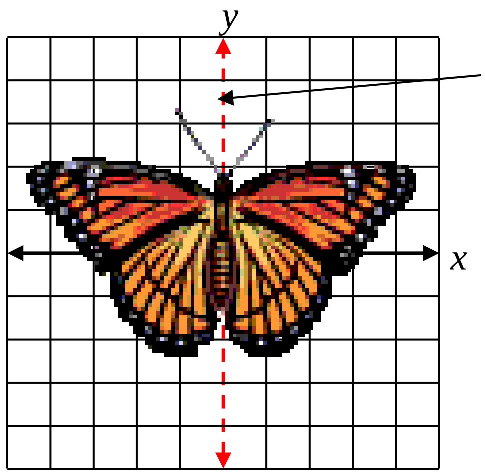
A **reflection** is a transformation of a shape in which the shape is reflected or flipped across a line. The line is called the **line of reflection**.

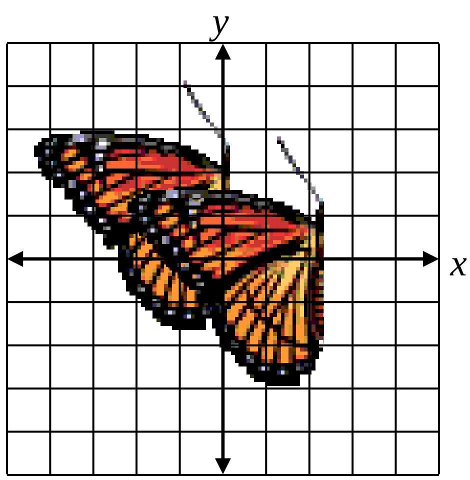


The duck is reflected in a pool of water creating a mirror image.

A figure has *line symmetry* if it can be folded over a line so that one half of the figure matches the other half. The line is called the **line of symmetry**.

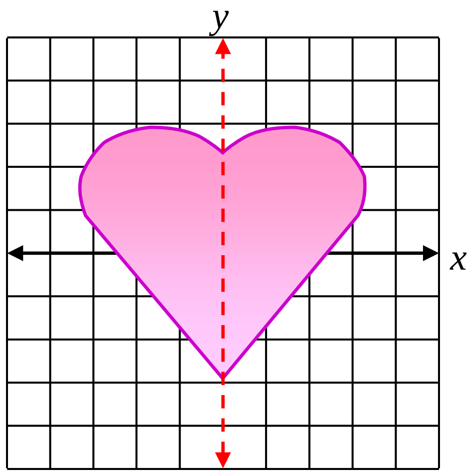
The **line of symmetry** divides the figure into two parts that are reflections of each other. The two sides are mirror images of each other. The structure of a butterfly is a natural example where line symmetry enhances its beauty.





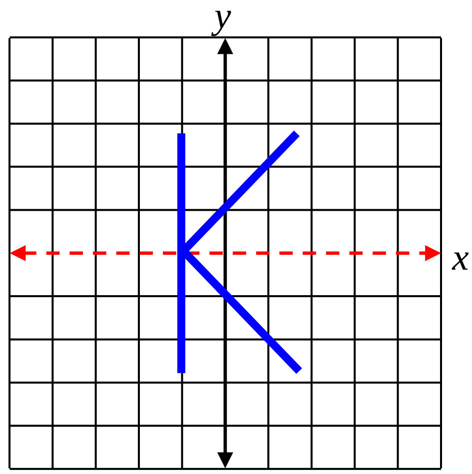
The left side of the butterfly is symmetrical to the right side. Symmetrical objects have parts that are congruent. The wings are congruent.

Samples of Line Symmetry



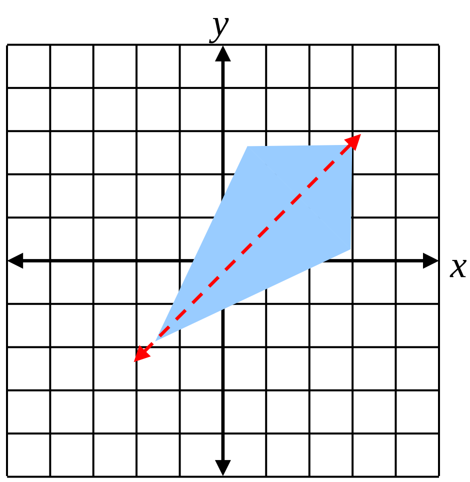
Vertical Line Symmetry

The left side of the heart is a reflection of the right side.



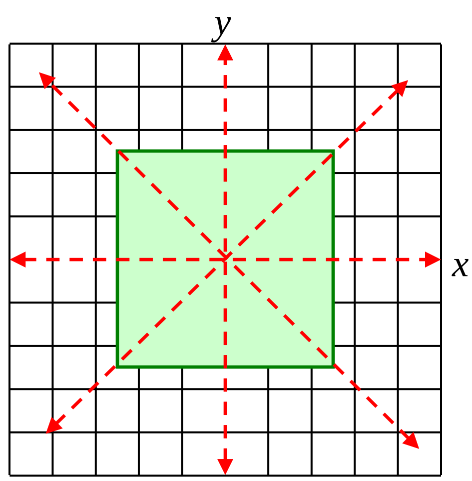
Horizontal Line Symmetry

The top part of the K is a reflection of the bottom part.



Diagonal Line Symmetry

The diagonal line cuts the kite into two symmetrical parts.



Four Lines of Symmetry

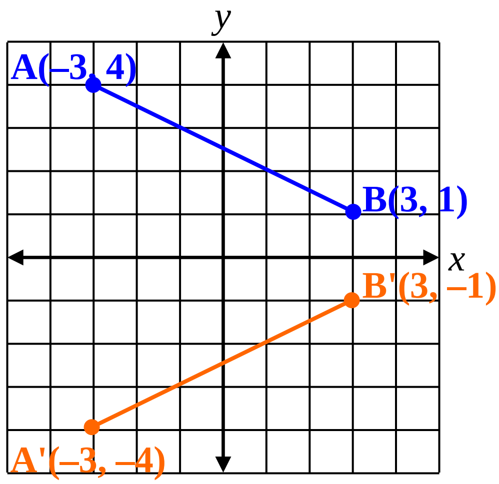
In this figure, there are four lines of symmetry; that is, four lines that cut the square into two symmetrical parts.

Notation in the Coordinate Plane: Reflections may be described using coordinate notation.

Reflection over the *x*-axis

A reflection over the *x*-axis occurs when the *y*-coordinate of an ordered pair is multiplied by –1. The line of symmetry is the *x*-axis.

Example 1: Reflect segment AB with endpoints A(–3, 4) and B(3, 1) over the *x*-axis.



Multiply the y-coordinate by -1.

$P(x, y) \rightarrow P'(x, -y)$

$A(-3, 4) \rightarrow A'(-3, -4)$

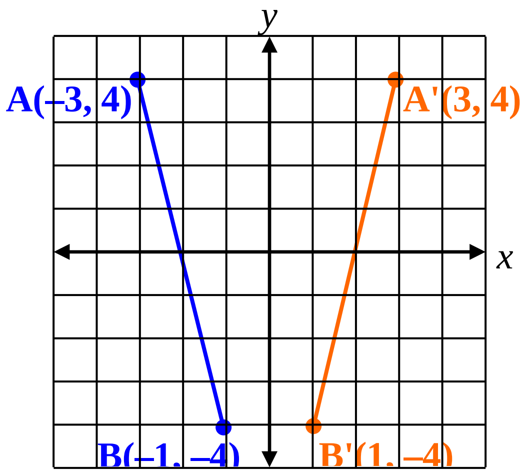
$B(3, 1) \rightarrow B'(3, -1)$

The endpoints of the reflected segment are $A'(-3, -4)$ and $B'(3, -1)$.

Reflection over the *y*-axis

A reflection over the *y*-axis occurs when the *x*-coordinate of an ordered pair is multiplied by –1. The line of symmetry is the *y*-axis.

Example 2: Reflect segment AB with endpoints A(–3, 4) and B(–1,–4) over the *y*-axis.



Multiply the x-coordinate by -1.

$P(x, y) \rightarrow P'(-x, y)$

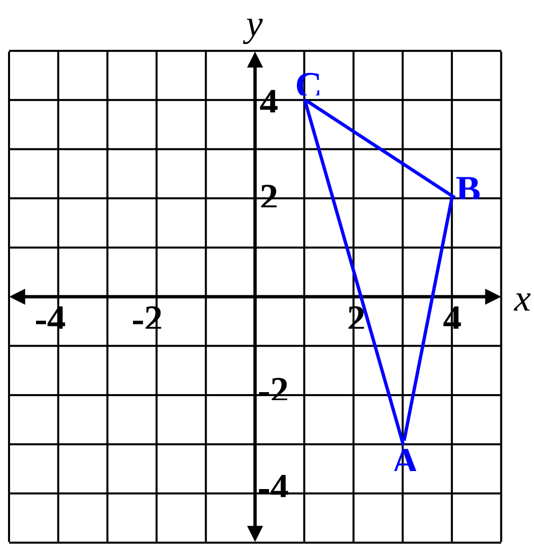
$A(-3, 4) \rightarrow A'(3, 4)$

$B(-1, -4) \rightarrow B'(1, -4)$

The endpoints of the reflected segment are $A'(3,4)$ and $B'(1,-4)$.

Example 3: Draw triangle ABC with vertices $A(3,-3)$, $B(4,2)$, and $C(1,4)$. Then find the coordinates of the vertices of the image after a reflection over the *y*-axis.

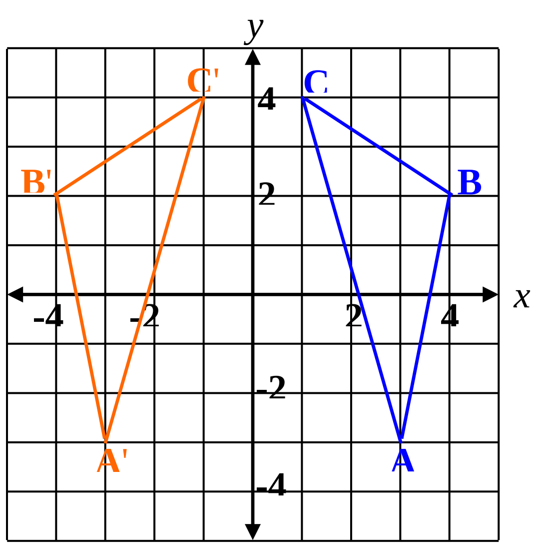
Step 1: Graph triangle ABC.



Step 2: Since the reflection is to made over the *y*-axis, multiply each *x*-coordinate by –1 to find the coordinates of the vertices of the image.

Original		Image
$P(x, y)$	$\rightarrow$	$P'(-x, y)$
$A(3, -3)$	$\rightarrow$	$A'(-3, -3)$
$B(4, 2)$	$\rightarrow$	$B'(-4, 2)$
$C(1, 4)$	$\rightarrow$	$C'(-1, 4)$

Step 3: Draw the image $A'B'C'$.

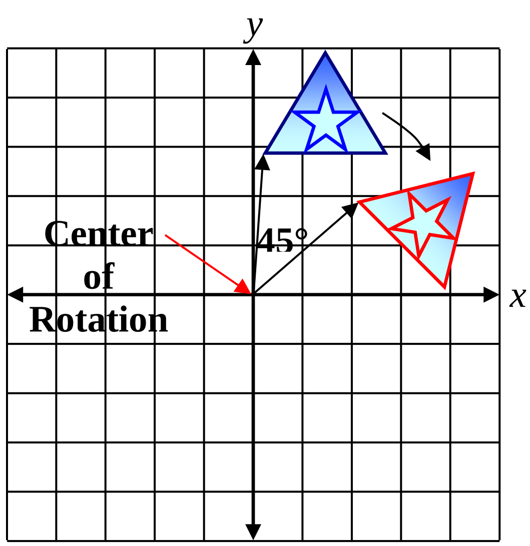


Triangle $A'B'C'$ is a reflection of triangle ABC. The *y*-axis is the line of symmetry.

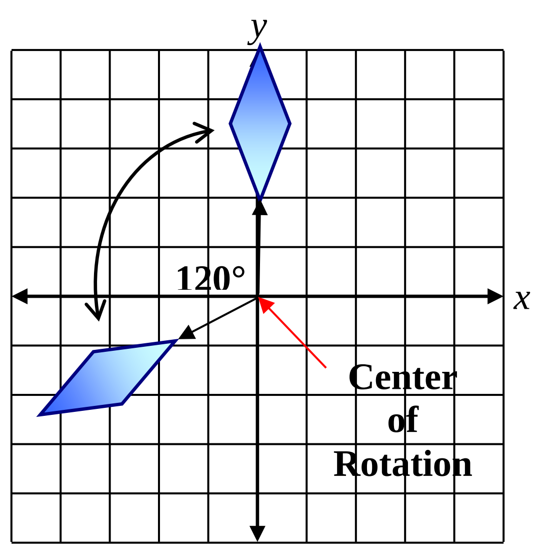
Rotations

A **rotation** is a transformation in which a figure is turned about a fixed point called the **center of rotation**.

The **angle of rotation** is an angle formed by rays drawn from the center of rotation through corresponding points on an original figure and its image. The direction of rotation can be *clockwise* or *counterclockwise*. The figure and its image are congruent.



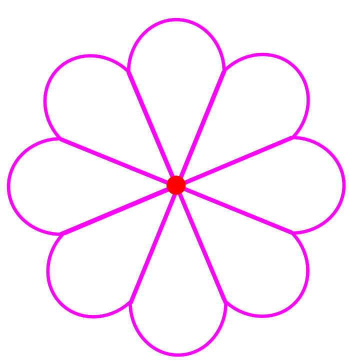
45° clockwise rotation

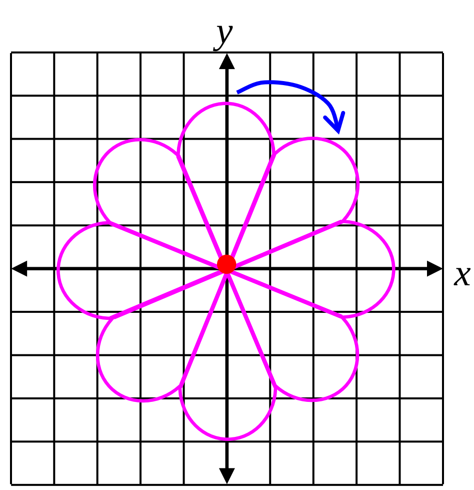


120° counterclockwise rotation

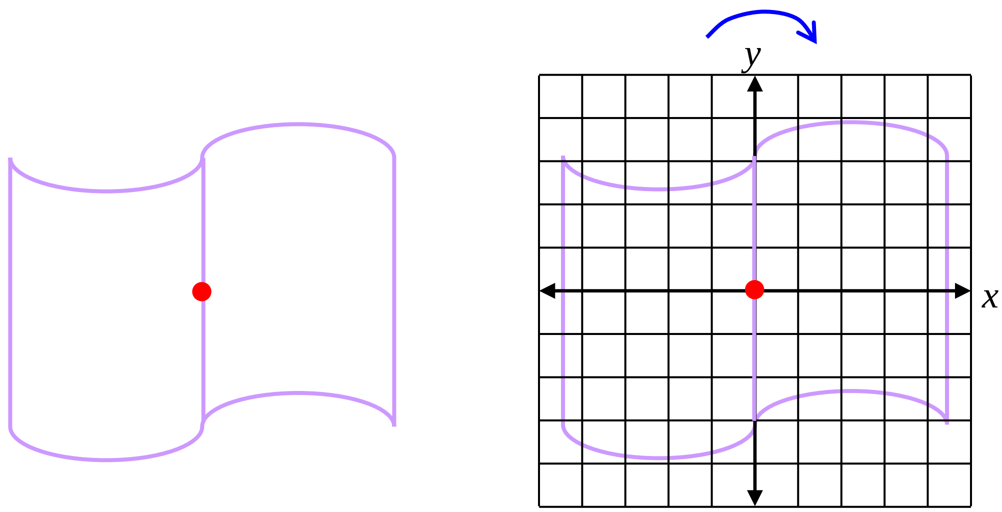
A figure has **rotational symmetry** if the figure can be rotated around some point so that it coincides with itself. The point is the center of rotation, and the amount of rotation must be less than one full turn; that is, less than 360°.

If you turn this figure around the center point, it coincides with itself every rotation of 45°, so it has rotational symmetry.





If you turn this figure around the given point of rotation, it coincides with itself every rotation of 180°, so it has rotational symmetry.



Notation in the Coordinate Plane: Rotations of each point (*x*, *y*) of a figure can be represented using coordinate notation.

Ninety-degree Clockwise Rotation

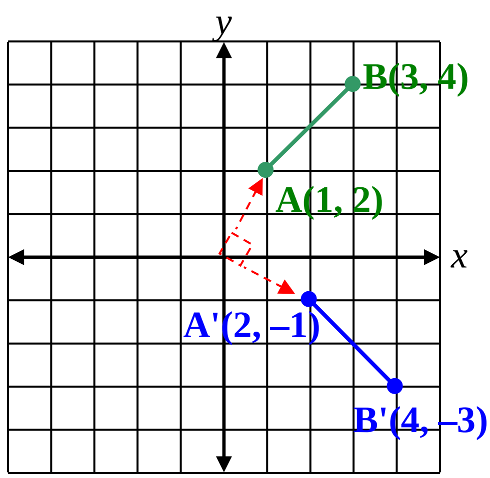
A 90-degree clockwise rotation occurs about the origin if the coordinates of the ordered pair are switched and the new *y*-coordinate is multiplied by –1.

Example 1: Rotate segment AB with endpoints A(1, 2) and B(3,4) about the origin with a 90-degree clockwise rotation.

To rotate segment AB ninety degrees, switch the coordinates, and then multiply the new *y*-coordinate by –1.

$$P(x, y) \rightarrow P'(y, -x)$$
$$(1, 2) \rightarrow (2, -1)$$
$$(3, 4) \rightarrow (4, -3)$$

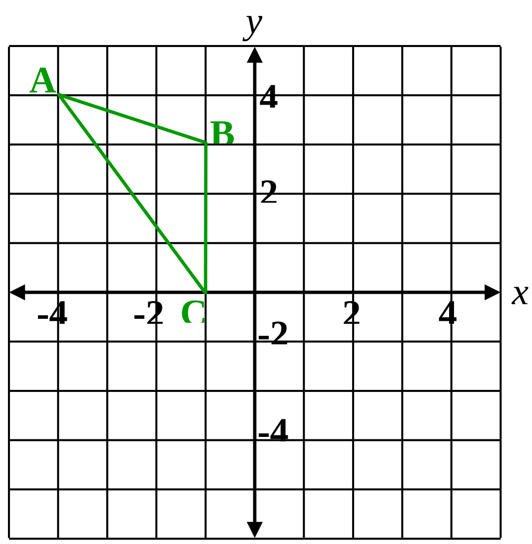
Graph the original segment (AB) and its image $(A'B')$.



The endpoints of the rotated segment are $A'(2, -1)$ and $B'(4, -3)$.

Example 2: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a **90°** clockwise rotation around the origin. Draw the image, A'B'C'.

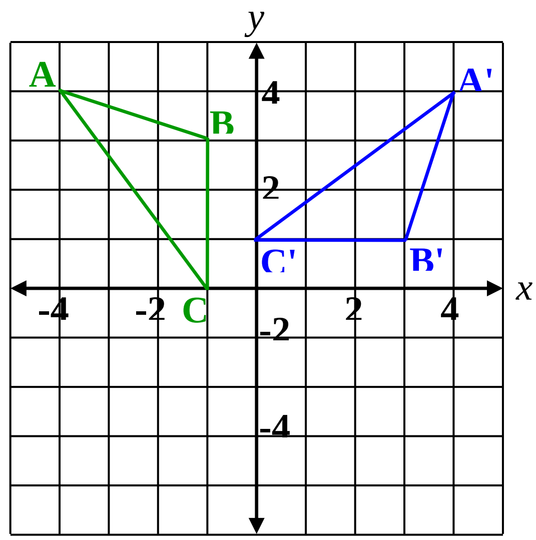
Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).



Step 2: Find the coordinates of the vertices of the image. Switch the coordinates and multiply the new *y*-coordinate by –1.

Original		Image
$P(x, y)$	$\to$	$P'(y, -x)$
$A(-4, 4)$	$\to$	$A'(4, 4)$
$B(-1, 3)$	$\to$	$B'(3, 1)$
$C(-1, 0)$	$\to$	$C'(0, 1)$

Step 3: Draw the image, A'B'C'.



Triangle *A'B'C'* is a 90-degree clockwise rotation of triangle *ABC*.

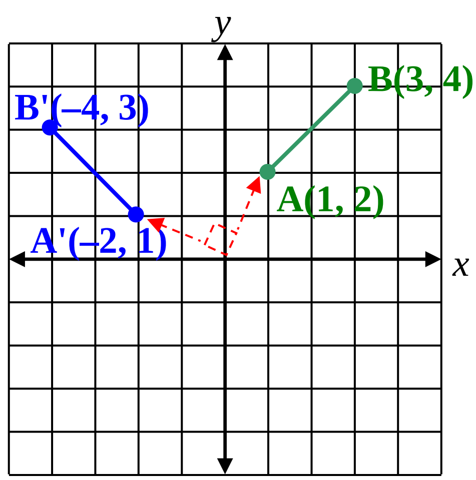
Ninety-degree Counterclockwise Rotation

A 90-degree counterclockwise rotation occurs about the origin if the coordinates of the ordered pair are switched and the new *x*-coordinate is multiplied by –1.

Example 3: Rotate segment AB with endpoints A(1, 2) and B(3,4) about the origin with a 90-degree counterclockwise rotation.

To rotate segment AB ninety degrees, switch the coordinates, and then multiply the new *x*-coordinate by –1.

$$
\begin{array}{lcl}
P(x,y) & \rightarrow & P'(-y,x) \\
(1,2) & \rightarrow & (-2,1) \\
(3,4) & \rightarrow & (-4,3)
\end{array}
$$



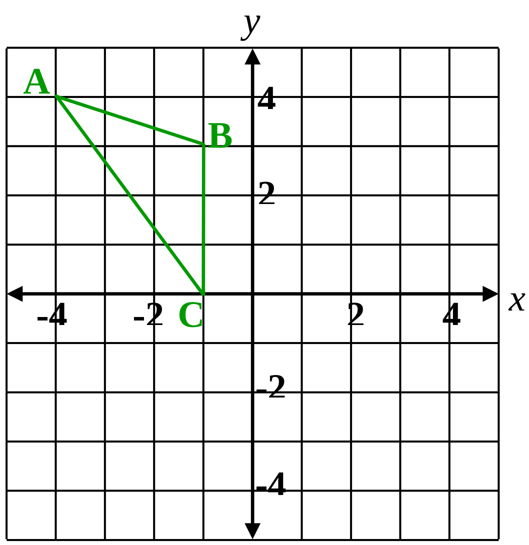
The endpoints of the rotated segment are $A'(-4,3)$ and $B'(-2,1)$.

180-Degree Rotation (Clockwise or Counterclockwise)

A 180-degree rotation occurs about the origin if the coordinates of the ordered are both multiplied by –1.

Example 4: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a **180°** rotation around the origin. Note: The image is the same whether the figure is rotated clockwise or counterclockwise. Draw the image, A'B'C'.

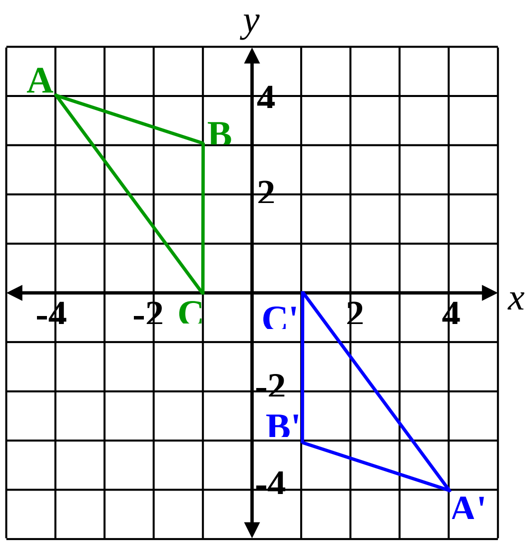
Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).



Step 2: Find the coordinates of the vertices of the image. Multiply both the *x*- and *y*-coordinates by –1.

Original		Image
$P(x, y)$	$\rightarrow$	$P'(-x, -y)$
$A(-4, 4)$	$\rightarrow$	$A'(4, -4)$
$B(-1, 3)$	$\rightarrow$	$B'(1, -3)$
$C(-1, 0)$	$\rightarrow$	$C'(1, 0)$

Step 3: Draw the image, A'B'C'.



Triangle $A'B'C'$ is a 180-degree rotation of triangle ABC .

Dilations

A **dilation** is a transformation in which the size is changed but not the shape. A dilation can be an enlargement or a reduction of a figure. The dilation of a figure is similar to the original image.

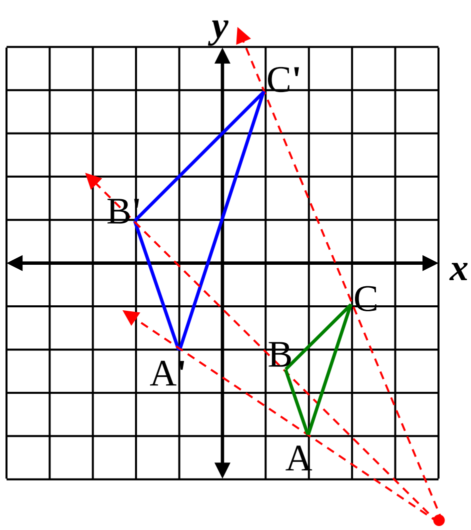
A dilation has a fixed point that is the **center of dilation**. The figure stretches or shrinks with respect to the **center of dilation**. To find the center of dilation, draw a line that connects each pair of corresponding vertices.

The **scale factor** describes how much a figure is enlarged or reduced. It is the ratio of the side length of the image to the corresponding side length of the original figure. A scale factor can be expressed as a decimal, a fraction, or a percent.

A **dilation** produces an image similar to the original figure.

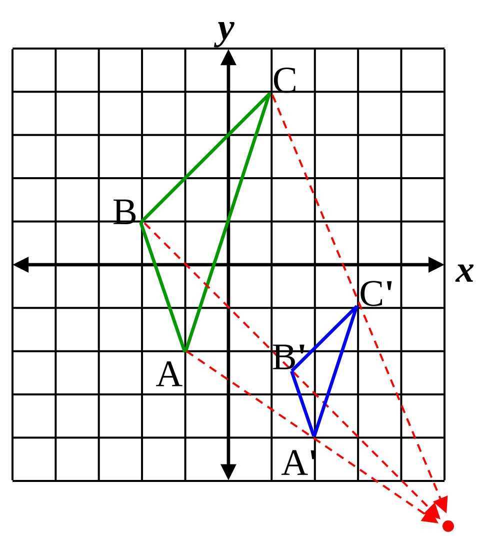
Multiplying by a scale factor > 1 ⟶ Enlarges a figure

Multiplying by a scale factor < 1 ⟶ Reduces a figure



Center of Dilation

Triangle A'B'C' is a dilation of triangle ABC and has a 100% increase in size.
The *scale factor* is 2.



Center of Dilation

Triangle A'B'C' is a dilation of triangle ABC and has a 100% decrease in size.
The *scale factor* is 0.5.

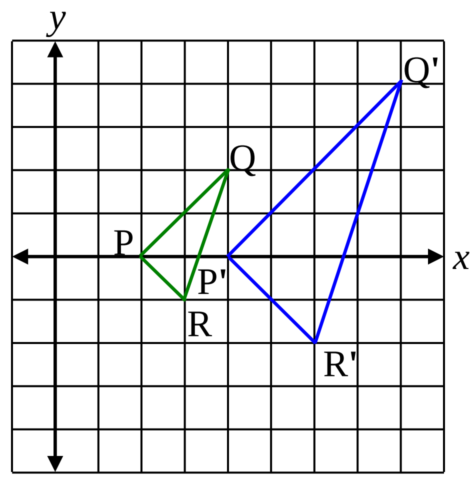
Example 1: Draw triangle PQR with vertices of P(2, 0), Q(4, 2), and R(3, –1). Then find the coordinates of the vertices of the image after the dilation having a scale factor of 2. Draw the image, P'Q'R'.

Step 1: Draw triangle PQR with vertices of P(2, 0), Q(4, 2), and R(3, –1).

Step 2: Find the coordinates of the vertices of the image. To dilate triangle PQR, multiply the *x*- and *y*-coordinates of each vertex by 2.

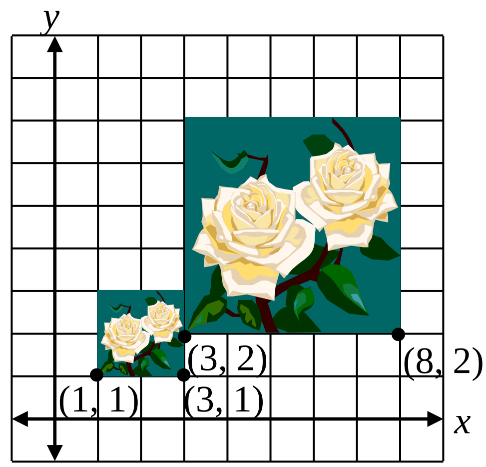
Original	**Image**
P(*x*, *y*)	*P*' (2*x*, 2*y*)
P(2, 0)	P'(4, 0)
Q(4, 2)	Q'(8, 4)
R(3, –1)	R'(6, –2)

Step 3: Draw the image, P'Q'R'.



Triangle P'Q'R' has a 100% increase.
The scale factor is 2.

Example 2: An artist uses a computer program and a coordinate grid to enlarge a design. Find the scale factor of the design.



Step 1: Find the width of the original design.

The width of the original design is the difference of the original design's *x*- coordinates.

(1,1) (3,1) $\rightarrow$ 3 – 1 = 2 units

Step 2: Find the width of the image.

The width of the enlargement is the difference of the enlargement's *x*- coordinates.

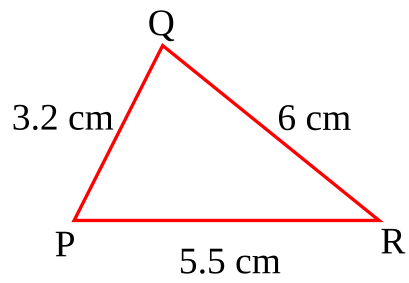
(3,2) (8,2) $\rightarrow$ 8 – 3 = 5 units

Step 3: Find the scale factor.

$$\text{Scale Factor } = \frac{\text{image}}{\text{original design}} = \frac{5}{2} = 2.5$$

The scale factor is 2.5.

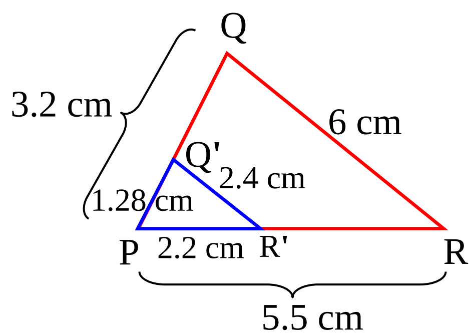
Example 3: Dilate triangle PQR by a scale factor of 0.4. Let vertex P be the center of dilation. Segment PR has a length of 5.5 centimeters, segment PQ has a length of 3.2 centimeters and segment QR has a length of 6 centimeters. P and P' are the same point.



Step 1: Multiply each side by 0.4.

Original	**Image**
(Length)	(0.4 × Length)
PR = 5.5 cm	PR' = 2.2 cm
PQ = 3.2 cm	PQ' = 1.28 cm
QR = 6 cm	Q'R' = 2.4 cm

Step 2: Draw the image, PQ'R'.



Triangle PQ'R' is a dilation of triangle PQR by a scale factor of 0.4 with point P as the center of dilation.