BETWEEN THE SET OF ORDERED PAIRS

In this unit, several important formulas related to coordinate geometry are presented. The midpoint formula is used to find the midpoint of a line segment. The Pythagorean Theorem is used to determine the distance between two points. It is also the basis for deriving the distance formula, a formula that is generally used when computing the distance between two points in the coordinate plane.

Midpoint in the Coordinate Plane

Pythagorean Theorem and the Distance Formula

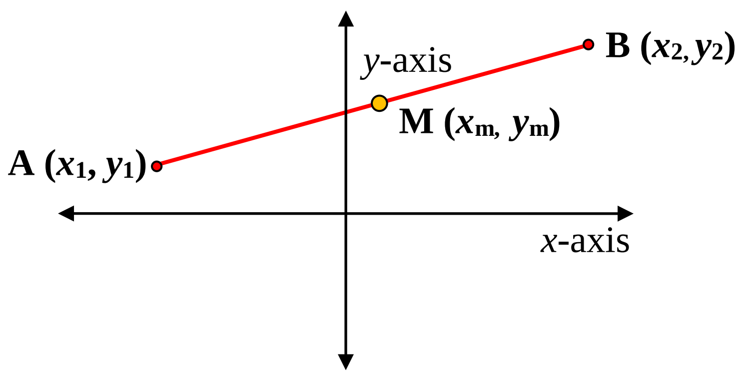
Midpoint in the Coordinate Plane

Midpoint Formula Coordinate Plane

In the coordinate plane, the coordinates of the midpoint of a segment with endpoints (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

To find the midpoint between two points in the coordinate plane, find the average distance between the two points. Add the *x*-coordinates and divide by two; and also add the *y*-coordinates and divide by two.

$$\mathbf{M} = (x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Example: Find the midpoint (M) of $\overline{AB}$ with endpoints A(–3, –2) and B(3, 4).

Let point A be (x_1, y_1) and point B be (x_2, y_2).

$$\text{Point A} \rightarrow (\underset{\displaystyle (x_1}{\underset{\downarrow}{-3}}, \underset{\displaystyle y_1)}{\underset{\downarrow}{-2}}) \qquad \text{Point B} \rightarrow (\underset{\displaystyle (x_2}{\underset{\downarrow}{3}}, \underset{\displaystyle y_2)}{\underset{\downarrow}{4}})$$

$$x_1 = -3, y_1 = -2 \qquad x_2 = 3, y_2 = 4$$

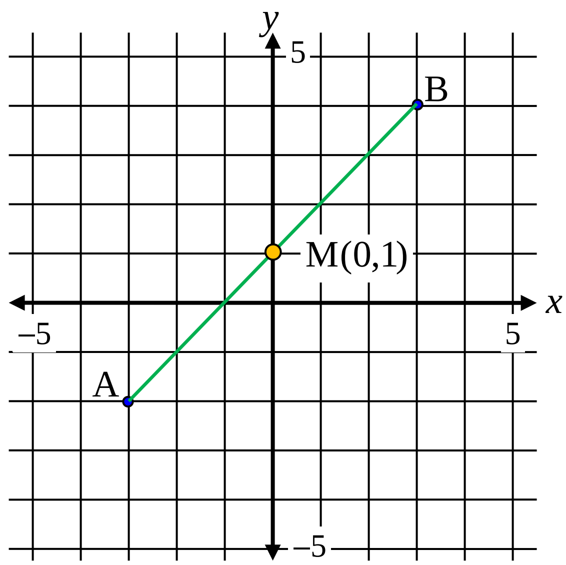
Now, substitute these values into the midpoint formula to find the midpoint.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M = \left(\frac{-3 + 3}{2}, \frac{-2 + 4}{2}\right)$$

$$M = \left(\frac{0}{2}, \frac{2}{2}\right)$$

$$M = (0,1)$$

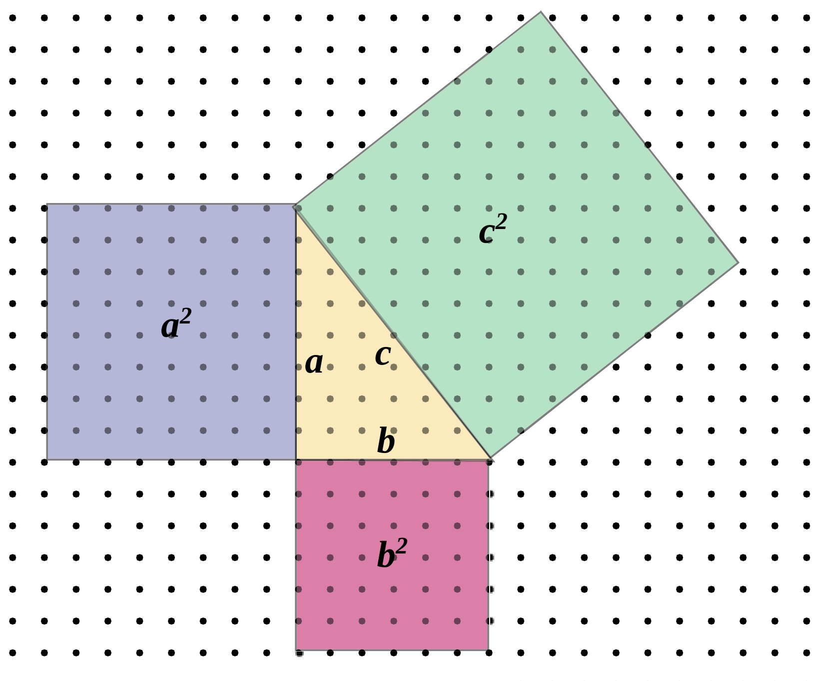


The midpoint of segment AB is point M(0, 1).

Pythagorean Theorem and the Distance Formula

theorem - A theorem is a mathematical statement that must be proven before it is accepted as being true.

Pythagorean Theorem - The Pythagorean Theorem is a relationship between the three sides of a right triangle. The sum of the squares of the two legs (the sides of the right triangle that make up the right angle) are equal to the square of the third side, the hypotenuse (the side opposite the right angle).



Special names are given to the sides of a right triangle. The two sides that make up the right triangle are called **legs** and the side opposite the right angle is called the **hypotenuse**.

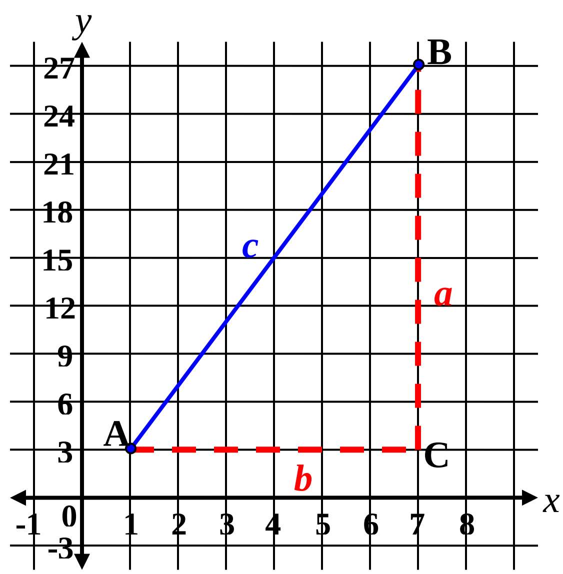
A special relationship exists between the sides of a right triangle. The sum of the squares of the two legs equals the square of the hypotenuse as shown in the figure above.

$$c^2 = a^2 + b^2$$

In the example above, the legs measure 6 and 8 units. What is the length of the hypotenuse?

$$c^2 = 6^2 + 8^2$$
$$c^2 = 36 + 64$$
$$c^2 = 100$$
$$c = \sqrt{100}$$
$$c = 10$$

Example 1: Use the Pythagorean Theorem to find the distance from A(1, 3) to B(7, 27).



Notice that the length of leg b (AC) is 7 – 1 = 6 units.

Notice that the length of leg a (BC) is 27 – 3 =24 units.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$6^2 + 24^2 = c^2$	Substitution
$36 + 576 = c^2$	Simplify
$612 = c^2$	Simplify
$\sqrt{612} = \sqrt{c^2}$	Take the square root of both sides of the equation.
$\pm\sqrt{612} = c$	Simplify $\sqrt{c^2} = \sqrt{c \cdot c} = c$.
$c \approx 24.7$	*When dealing with length, we consider only the positive root.

The hypotenuse is approximate 24.7 units in length.

* $\sqrt{612}$ is an irrational number and its root extends on forever, so for our purposes in this course, we round the answer. The symbol for approximately equal is $\approx$. We ignore the negative root as we are concerned only about distance in this problem.

Now, let's revisit this problem and make some observations.

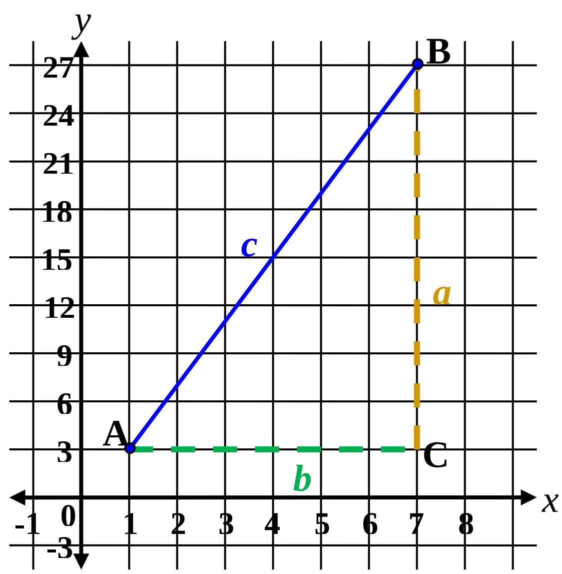
(1) The vertices of the right triangle that can be made with AB are:

Point A (1, 3) Point B (7, 27)   Point C (7, 3)

(2) The length of BC (a) is the difference in the y-coordinates of point B and point C. (This leg is parallel to the y-axis.)

Point B (7, 27) Point C (7, 3)

$BC = |\,27 - 3| = |24| = 24$



(3) The length of AC (b) is the difference in the x-coordinates of point A and point C. (This leg is parallel to the x-axis.)

Point A (1, 3) Point C (7, 3)

$AC = |\,1 - 7| = |-6| = 6$

*Absolute value notation is used when working with lengths so that the differences will be positive values.

The distance formula is based on these observations. Let's take a look at the formula to compare.

Distance Formula

The distance d between any two points with coordinates (x_1, y_1) and (x_2, y_2) is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **.**

This formula can be interpreted as follows:

To find the distance between any two points in the coordinate plane,

(a) Find the difference of the x-coordinates and square them

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(b) Find the difference of the y-coordinates and square them

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

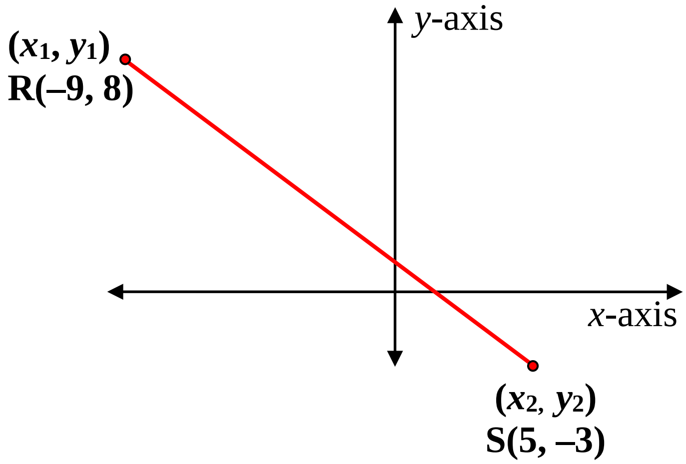
(c) Add these two squares

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

(d) Find the square root of the sum.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2: Find the length of $\overline{RS}$ between points R(–9, 8) and S(5, –3).



Let point R be (x_1, y_1) and point S be (x_2, y_2).

$$\text{Point R} \to (-9, 8) \quad \downarrow \downarrow \quad (x_1, y_1) \qquad x_1 = -9, y_1 = 8$$

$$\text{Point S} \to (5, -3) \quad \downarrow \downarrow \quad (x_2, y_2) \qquad x_2 = 5, y_2 = -3$$

Now, substitute into the distance formula to find the length of RS.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{[5 - (-9)]^2 + (-3 - 8)^2}$$

$$d = \sqrt{(14)^2 + (-11)^2}$$

$$d = \sqrt{196 + 121}$$

$$d = \sqrt{317}$$

$$d \approx 17.8$$

The length of RS is approximately 17.8 units.