DERIVE FORMULAS

In this unit, you will revisit some of the formulas mentioned in previous units. You will take a closer look at how they are developed by deriving the volume formulas for a cone and a pyramid and the surface area formulas of a rectangular prism and a square pyramid. You will also examine solutions and determine if an answer is reasonable.

Derive the Formula for:

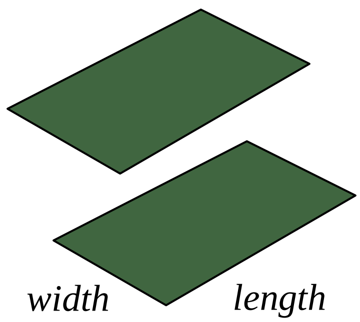
- Surface Area of a Rectangular Prism
- Surface Area of a Pyramid
- Volume of a Cone and Volume of a Pyramid

Determine Reasonableness of Solutions

Derive the Formula for the Surface Area of a Rectangular Prism

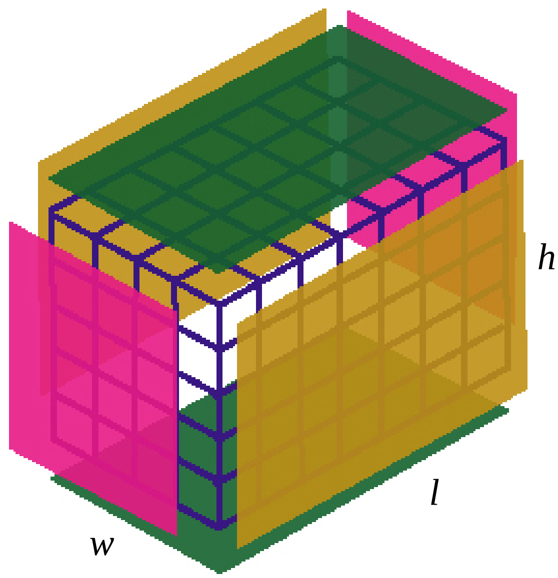
When considering the surface area of different shapes, we can keep track of the shapes that make up the faces of the solid. That will help us to take a short cut when calculating the total surface area.

The surface area of a rectangular prism can be found as follows.



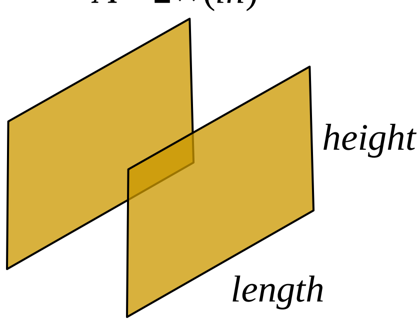
Area of top rectangle
and bottom rectangle
are the same.

$$A = 2 \times (lw)$$



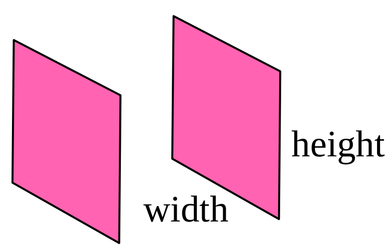
Area of front rectangle
and back rectangle
are the same.

$$A = 2 \times (lh)$$



Area of left rectangle
and right rectangle
are the same.

$$A = 2 \times (wh)$$



Notice how each calculation was multiplied by two. This occurred three times which would account for all six sides.

Add to find the total surface area.

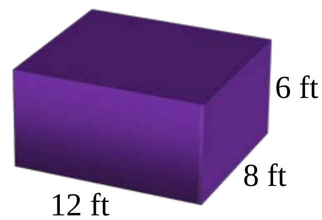
Now let's make a formula for finding the surface area of a rectangular prism letting *T* represent the surface area:

$$T = 2(lw) + 2(lh) + 2(wh)$$

This formula will work for any rectangular prism.

Example: Find the surface area of a rectangular prism that measures 12 feet by 8 feet by 6 feet.

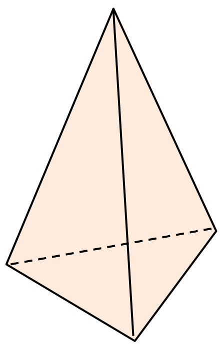
$$T = 2(lw) + 2(lh) + 2(wh)$$
$$T = 2(12)(8) + 2(8)(6) + 2(12)(6)$$
$$T = 192 + 96 + 144$$
$$T = 432$$



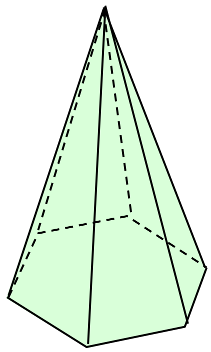
The surface area of the rectangular prism is 432 square feet (ft^2).

Derive the Formula for the Surface Area of a Pyramid

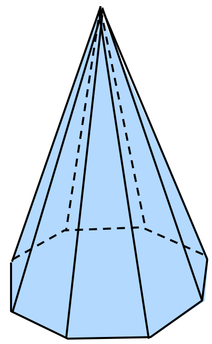
If we look at different views of a pyramid, there are a few characteristics we can note. Namely, the sides are always triangles, but the base can be any shape.



Triangular Pyramid



Hexagonal Pyramid

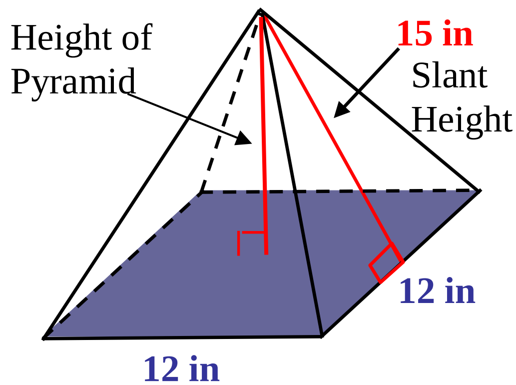


Octagonal Pyramid

If we calculate the surface area of several different pyramids, we discover that we can simplify the process by using a formula. We will examine the development of a formula for a pyramid with a square base given that the edge of the square base measures 12 inches and the slant height of the triangular side measures 15 inches.

Square Pyramid

A square pyramid is a pyramid with square base.



First calculate the area of the square on the bottom.

We will refer to the edge of the square as the base (*e*).

$A = e^2$	Formula for area of a square.
$A = 12^2$	Substitute 12 for *e*.
$A = 144 \text{ in}^2$	Simplify.

Next calculate the area of one triangular side.

Let *b* and *h* represent the base and height of the triangle, respectively.

$A = \frac{1}{2} \times b \times h$	Formula for area of a triangle.
$A = \frac{1}{2} \times 12 \times 15$	Substitute base and height dimensions.
$A = 90 \text{ in}^2$	Simplify.

*Note: This calculation uses the slant height, not the height of the pyramid.

Now consider all four triangles.

$$4A = 4(90) = 360 \text{ in}^2$$

Finally, add the areas of the sides and the area of the bottom.

$$\begin{aligned} \text{Surface Area} &= e^2 + 4A \\ &= 144 + 360 \\ &= 504 \end{aligned}$$

The surface area of the square pyramid is 504 square inches (in^2).

Now let's put a formula together. The formula for the surface area (*T*) of a square pyramid is:

$$T = e^2 + 4\left[\frac{1}{2}b \times h\right]$$

This formula can be used to find the surface area of any square pyramid.

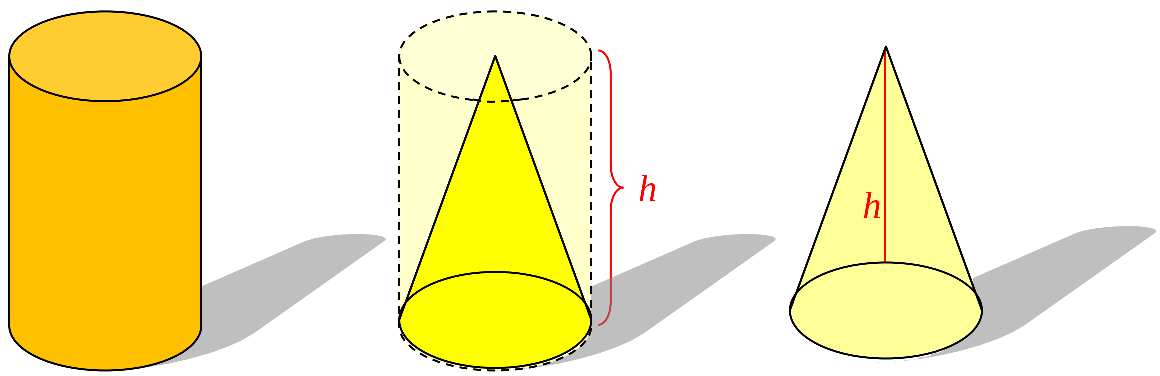
We can certainly use these formulas to find the surface area and we can develop our own as we go along.

Derive the Formula for the Volume of a Cone and the Volume of a Pyramid

In earlier units we used the formula for calculating the volume of a cylinder or the volume of a rectangular prism. You can use this idea with one modification to find the volume of a pyramid or cone. This modification has been discovered by experimentation over many years. The idea is simple. The volume of a pyramid or a cone is **one-third** of the volume of a cylinder or a rectangular prism with the same height.

Cone

A cone's volume can be found in a similar way as finding the volume of a cylinder. The volume of a cone is equal to 1/3 of the volume of a cylinder with the same base area and height.



$V = \pi r^2 h$ $V = \frac{1}{3}$ of Cylinder $V = \frac{1}{3}\pi r^2 h$

Example 1: Find volume of a cone with a radius of five (5) inches and a height of nine (9) inches.

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3} \times 3.14 \times 5^2 \times 9$$

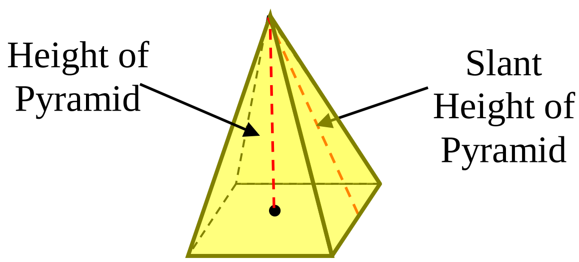
$$V = \frac{1}{3} \times 3.14 \times 25 \times 9$$

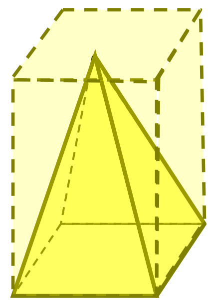
$$V = 235.5$$

The volume of the cone is 235.5 cubic inches (in^3).

Pyramid

We can find the volume of a pyramid in a similar manner as finding the volume of a cone. Be sure to use the height of the pyramid, not the slant height, and use the area of the base of the pyramid. The volume of a pyramid is 1/3 the volume of a rectangular prism with the same base area and height.





$$V = \frac{1}{3}Bh$$

Example 2: Find volume of a pyramid with a square base edge that measures eight (8) feet and a height that measures 12 feet.

$$V = \frac{1}{3}Bh$$

$$V = \frac{1}{3}\times(8\times8)\times12$$

$$V = \frac{1}{3}\times64\times12$$

$$V = 256$$

The volume of the pyramid is 256 cubic feet (ft^3).

Determine the Reasonableness for Solutions

First let's consider this formula, $d = rt$, read "*distance* = *rate* × *time*".

> *Example 1*: Find the distance traveled if a car traveled 50 miles per hour for 30 minutes.

$$\begin{aligned} d &= \quad r \quad \times \quad t \\ d &= \quad 50 \text{ mph} \quad \times \quad 30 \text{ min} \\ d &= \quad 1500 \text{ miles} \end{aligned}$$

This problem seems okay, but if this were accurate, a trip from Pittsburgh to New York City, approximately 500 miles would take 10 minutes ($50 \times 10 = 500$). Remember this is a car not an super sonic jet!

Let's see what went wrong!

distance = miles rate = miles per HOUR or $\dfrac{\text{miles}}{\text{hour}}$

Time = 30 MINUTES (aha!)

The time should be $\dfrac{1}{2}$ hour, not 30 minutes; because 30 minutes is $\dfrac{30}{60}$ or $\dfrac{1}{2}$ hour. (There are 60 minutes in an hour.)

*The units must be in **agreement**. If the rate is miles PER HOUR, then the time must be expressed in HOURS.

Example 1(revisited): Find the distance traveled if a car traveled 50 miles per hour for 30 minutes.

A better calculation is:

Change 30 minutes to $\frac{1}{2}$ hour, then calculate the distance.

$$d = r \times t$$

$$d = 50 \text{ mph} \times \frac{1}{2} \text{ hour}$$

$$d = 25 \text{ miles}$$

Now let's check the reasonableness of this answer by thinking about how long the trip from Pittsburgh to New York City will take.

25 miles in 1/2 hour
50 miles in 1 hour
500 miles in 10 hours

Ten hours is a more reasonable estimate for the time it takes to travel in a car from Pittsburgh to New York City.

Now let's make sure we will avoid this common mistake.

Since 50 mph is a unit of measure that is miles per hour, this requires that all values in the problem be the same units. Thirty minutes must be adjusted to hours (1/2 hour).

*The units must be in **agreement**. If the rate is miles PER HOUR, then the time must be expressed in HOURS.

Now we'll try a problem where we previously derived the formula for finding the surface area of a square pyramid.

Example 2: Find the area of a square pyramid with a base edge of 30 centimeters and a slant height of 400 millimeters.

*The units must be in **agreement**. The base edge is given in CENTIMETERS and the slant height is given in MILLIMETERS.

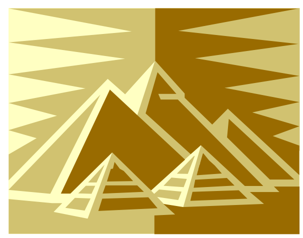
We'll adjust the 400 millimeters to 40 centimeters.
(10 mm = 1 cm, so 400 $\div$10 = 40).

$e = 30$ $b = 30$ $h = 40$

$$T = e^2 + 4\left[\frac{1}{2}b \times h\right]$$

$$T = (30)^2 + 4\left[\frac{1}{2}(30)(40)\right]$$

$$T = 3{,}300 \text{ cm}^2$$



Now we have a reasonable answer!