SEQUENCES AND SERIES

An important mathematical skill is discovering patterns. In this unit you will investigate different types of patterns represented in sequences.

Sequence

Series

Arithmetic Sequences

Arithmetic Series

Geometric Sequences

Geometric Series

Sequence

A **sequence** is an ordered set of numbers which is represented by a function. The terms of a sequence are generally arranged in a pattern. Some examples are shown below.

1, 3, 5, 7, …   –3, 0, 3, 6, 9, …

5, 25, 125, 625   2, 4, 6, 8

The three dots after the top two sequences indicate that they are **infinite**, continue without end. The bottom two sequences have no dots, which indicate they are **finite**, have a last term.

As stated above, an infinite sequence can be defined by a function, whose domain is the set of natural numbers {1, 2, 3, …*n*, …} and a finite sequence is defined by a function whose domain is the first n natural numbers {1, 2, 3, …*n*}

Example #1: The sequence 1, 3, 5, 7, … can be defined by the function *f*, shown below.

$f(n) = 2n - 1$, where n = 1, 2, 3, …

n	1	2	3	4	5	← domain
***f* (*n*)**	1	3	5	7	9	← range

The range value $f(n)$ is usually symbolized with a symbol such as a_n. Thus, in the equation from above, $f(n) = 2n - 1$, we would have $a_n = 2n - 1$. Each member of the range is a term of the sequence:

a_1 is the first term, a_2 is the second term, and a_n is the nth term

$a_1 = 2(1) - 1 = 1$ first term

$a_2 = 2(2) - 1 = 3$ second term

$a_3 = 2(3) - 1 = 5$ third term

. .
. .
. .

The ordered list of elements would be 1, 3, 5, $2n - 1$, …

Explicit Formula

An **explicit formula** is a formula that defines the *n*th term of a sequence. To find each term of a sequence using an explicit formula, substitute the number of the term for *n*.

Example #2: Write the first five terms of the sequence defined by the explicit formula $a_n = -2n + 2$.

a.) Evaluate $a_n = -2n + 2$ for *n*-values 1, 2, 3, 4, and 5

b.) $a_1 = -2(1) + 2 = 0$

$a_2 = -2(2) + 2 = -2$

$a_3 = -2(3) + 2 = -4$

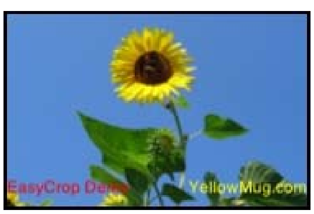
$a_4 = -2(4) + 2 = -6$

$a_5 = -2(5) + 2 = -8$

c.) The first five terms of the sequence $a_n = -2n + 2$ are 0, –2, –4, –6, and –8.

Recursion Formula

A **recursion formula** is a formula that defines each term in terms of one or more preceding terms. One such famous sequence that uses a recursion formula is the Fibonacci Sequence, which is used to describe a wide variety of structures in nature. One such structure is the spiral pattern of sunflower seeds.



Example #3: Write the first five terms of the sequence defined by the recursive formula $a_1 = 4$ and $a_n = 3a_{n-1} + 5$, where $n \geq 2$.

a.) Substitute *n* with 2 in $a_n = 3a_{n-1} + 5$

$$a_n = 3a_{n-1} + 5$$

$$a_2 = 3a_{2-1} + 5 = 3a_1 + 5$$

b.) Substitute a_1 with 4

$$a_2 = 3(4) + 5 = 17$$

c.) Now a_2 = 17, so substitute a_2 with 17

$$a_3 = 3a_{3-1} + 5 = 3a_2 + 5$$
$$a_3 = 3(17) + 5$$
$$a_3 = 56$$

d.) $a_3 = 56$ so substitute 56 for a_3

$$a_4 = 3a_{4-1} + 5 = 3a_3 + 5$$
$$a_4 = 3(56) + 5$$
$$a_4 = 173$$

e.) $a_4 = 173$ so substitute 173 for a_4

$$a_5 = 3a_{5-1} + 5 = 3a_4 + 5$$
$$a_5 = 3(173) + 5$$
$$a_5 = 524$$

We now have:

$$a_1 = 4$$

$$a_2 = 17$$

$$a_3 = 56$$

$$a_4 = 173$$

$$a_5 = 524$$

Therefore the first five terms of the sequence are 4, 17, 56, 173, and 524.

Series

A **series** is an expression that indicates the sum of terms of a sequence. For example, if $a_1,\ a_2,\ a_3,\ ...a_n...$ is a sequence, then $a_1 + a_2 + a_3 + ...\ a_n + ...$ is a series.

A series is represented in **summation notation** using the symbol $\sum$, the Greek letter sigma. Consider the following example.

Example #1: $\sum_{n=1}^{3} a_n = a_1 + a_2 + a_3$

The terms on the right $(a_1 + a_2 + a_3)$ are obtained by replacing the **summing index** *n* with integers starting with the first number indicated below $\sum$ and ending with the number above $\sum$.

Example #2: Write the terms of each series, and then evaluate.

a.) $\sum_{n=1}^{5} 3n =$ 3(1) + 3(2) + 3(3) + 3(4) + 3(5)

= 3 + 6 + 9 + 12 + 15

= 45

b.) $3\sum_{n=1}^{5} n =$ 3(1 + 2 + 3 + 4+ 5)

= 3(15)

= 45

Arithmetic Sequences

An arithmetic sequence is a sequence whose terms differ by the same number *d*, called the common difference. That is, if a_n, where $n \geq 2$, is any term in an arithmetic sequence, then:

$a_n - a_{n-1} = d$ or $a_n = a_{n-1} + d$ is true.

*n*th Term of an Arithmetic Sequence

The general term a_n, of an arithmetic sequence whose first term is a_1 and whose common difference is *d*, is given by the formula

$$a_n = a_1 + (n-1)d$$

Example #1: Find the fourth term of the sequence defined by the recursive formula $a_1 = -4$ and $a_n = a_{n-1} + 3$.

a.) The sequence is arithmetic in which $a_1 = -4$ and the common difference is 3.

b.) Use the general formula $a_n = a_1 + (n-1)d$ to find the fourth term by replacing n with 4 (because this is the term you are looking for)

$a_n = a_1 + (n-1)d$

$a_4 = -4 + (4-1)3$

$a_4 = -4 + (3)3$

$a_4 = -4 + 9$

$a_4 = 5$

Thus the fourth term of the sequence is 5.

Arithmetic Series

An arithmetic series is the indicated sum of the terms of an arithmetic sequence.

Consider the sequence 2, 4, 6, …The sum of the first five terms of this sequence is denoted S_5.

$$S_5 = 2 + 4 + 6 + 8 + 10$$

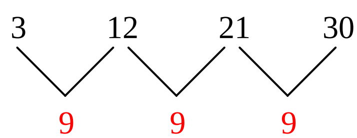
$$S_5 = 30$$

A formula can be used to find the sum of the first *n* terms of an arithmetic series. The sum, S_n, of the first *n* terms of an arithmetic series with first term a_1 and the *n*th term a_n is:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

Example #1: Given 3 + 12 + 21 + 30 + … find S_{23}

a.) Find the common difference



b.) $a_1 = 3$ and $d = 9$ (from above) find a_{23}

$$a_{23} = 3 + (23 - 1)9$$

$$a_{23} = 3 + (22)9$$

$$a_{23} = 201$$

c.) use $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ to find S_{23}

$$S_{23} = 23\left(\frac{3+201}{2}\right)$$

$$S_{23} = 23(102)$$

$$S_{23} = 2346$$

Summation notation can also be used to evaluate the sum of terms of an arithmetic series.

Example #2: Evaluate $\sum_{k=1}^{10}(8-2k)$

a.) This summation notation describes the summation of the first ten terms of the arithmetic series that begins

6 + 4 + 2 + 0 + …, in which $a_1 = 6$ (this is found be replacing *k* with 1 and evaluating the function) and *d* = (–2) (this is found by finding the difference between the terms of the sequence)

b.) First find a_{10} using $a_n = a_1 + (n-1)d$

$$a_n = a_1 + (n-1)d$$

$$a_{10} = 6 + (10-1)(-2)$$

$$a_{10} = 6 + (9)(-2)$$

$$a_{10} = -12$$

c.) Now we can find S_{10} using $S_n = n\left(\frac{a_1 + a_n}{2}\right)$

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

$$S_{10} = 10\left(\frac{6+(-12))}{2}\right)$$

$$S_{10} = 10\left(\frac{-6}{2}\right)$$

$$S_{10} = 10(-3)$$

$$S_{10} = -30$$

Geometric Sequences

A **geometric sequence** is a sequence in which the ratio of successive terms is the same number, *r*, called the **common ratio**. That is, if a_n is any term in a geometric sequence, then

$$\frac{a_n}{a_{n-1}} = r \text{, where } n \geq 2$$

This formula for the general term is a recursive formula.

*n*th Term of a Geometric Sequence

The *n*th term, a_n of a geometric sequence whose first term is a_1 and whose ratio is *r* is given by the explicit formula

$$a_n = a_1 r^{n-1} \text{, where } n \geq 1$$

Example #1: Find the sixth term of the sequence defined by the recursive formula $a_1 = 6$ and $a_n = 3a_{n-1}$

a.) This is a geometric sequence in which $a_1 = 6$ and $r = 3$.

b.) Use the explicit formula $a_n = a_1 r^{n-1}$ to find the sixth term.

$$a_6 = 6(3)^{6-1}$$

$$a_6 = 6(3)^5$$

$$a_6 = 6(243)$$

$$a_6 = 1458$$

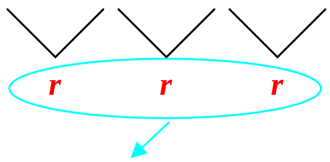
The sixth term of the sequence is 1458.

If you are given any two terms of a geometric sequence, then you can often write all the terms.

Example #2: Find the 10th term of the geometric sequence in which $a_2 = 48$ and $a_5 = 384$.

a.) Find the common ratio, *r*

n	2	3	4	5
a_n	48	?	?	384



$$48r^3 = 384$$

*Since there are 3 *r*'s between 48 and 384, this becomes the exponent on *r* so that you can solve for *r*.

$$48r^3 = 384$$

$$r^3 = 8$$

$$r = 2$$

b.) Find a_1 using $a_2 = 48$ or $a_5 = 384$

Using $a_2 = 48$:

$$a_n = a_1 r^{n-1}$$

$$a_2 = a_1 r^{2-1}$$

$$48 = a_1(2)^1$$

$$24 = a_1$$

Using $a_5 = 384$:

$$a_n = a_1 r^{n-1}$$

$$a_5 = a_1 r^{5-1}$$

$$384 = a_1(2)^4$$

$$384 = a_1(16)$$

$$24 = a_1$$

c.) Find the 10th term now that you know $a_1 = 24$ and r = 2 by using

$$a_n = a_1 r^{n-1}$$

$$a_{10} = 24(2)^{10-1}$$

$$a_{10} = 24(2)^9$$

$$a_{10} = 12{,}288$$

Geometric Series

A geometric series is the indicated sum of the terms of a geometric sequence. Consider the sequence 3, 9, 27, 81, … The sum of the first five terms, denoted S_5 is:

$$S_5 = 3 + 9 + 27 + 81 + 243 = 363$$

A formula can be used to find the sum of a geometric series.

The sum, S_n, of the first *n* terms of a geometric series is:

$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$, where a_1 is the first term, r is the common ratio and $r \neq 1$.

Example #1: Given 1 + 2 + 4 + 8 + … Find S_{11}.

a.) Find the common ratio $\frac{a_n}{a_{n-1}} = \frac{2}{1} = \frac{4}{2} = \frac{8}{4} = 2$

b.) Substitute 1 for a_1, 2 for r and 11 for n in $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

$$S_{11} = 1\left(\frac{1-2^{11}}{1-2}\right)$$

$$S_{11} = 1\left(\frac{-2047}{-1}\right)$$

$$S_{11} = 2047$$

Summation notation can be used to evaluate a geometric series.

Example #2: Evaluate $\sum_{n=1}^{7} 4(-5)^{n-1}$

a.) Use $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$

$$S_7 = 4\left(\frac{1-(-5)^7}{1-(-5)}\right)$$

$$S_7 = 4\left(\frac{1+78125}{6}\right)$$

$$S_7 = 4(13021)$$

$$S_7 = 52{,}084$$