MORE CONIC SECTIONS

In this unit you will study two more conic sections called an ellipse and hyperbola and learn how to write the standard form of each given certain information.

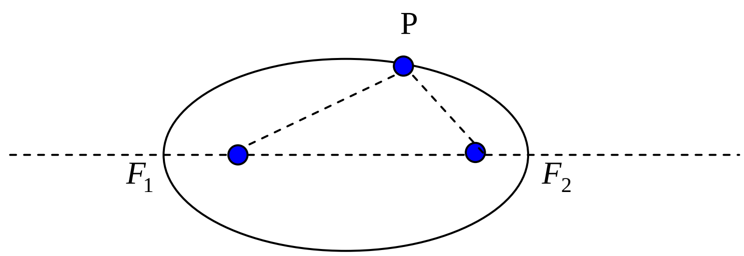
Ellipse

Hyperbola

Double-Concentric Graph Paper

Ellipse

Ellipse: the set of all points P in a plane such that the **sum** of the distances from P to two fixed points, F_1 and F_2 called the foci, is a constant.



Note: Point P is represented at the top; and the foci points, which are labeled, are the other two points.

The definition of ellipse is basically saying that:

the distance between P and F_1	+	the distance between P and F_2	=	a given constant

The standard form of an ellipse is given as:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \text{ with a center at } (h, k)$$

*An ellipse has two axes of symmetry

major axis: the longer axis

minor axis: the shorter axis

Writing the Equation of an Ellipse

If you are given the two foci and their constant sum, you are able to write the equation of an ellipse in standard form.

Example #1: Write the equation of an ellipse whose foci are $F_1(-4, 0)$ and $F_2(4, 0)$ and whose constant sum is 10. Use the definition of an ellipse to write the equation.

the distance between P and F_1	+	the distance between P and F_2	=	a given constant

$$\sqrt{(x+4)^2+(y-0)^2} \quad + \quad \sqrt{(x-4)^2+(y-0)^2} \quad = \quad 10$$

$$\sqrt{(x+4)^2+y^2} = 10 - \sqrt{(x-4)^2+y^2}$$

$$[\sqrt{(x+4)^2+y^2}\,]^2 = [(10-\sqrt{(x-4)^2+y^2})(10-\sqrt{(x-4)^2+y^2})]$$

Square both sides and **FOIL** the right side of the equation. Also expand any square quantities.

$$(x+4)^2+y^2 = 100-10\sqrt{(x-4)^2+y^2} - 10\sqrt{(x-4)^2+y^2} + [(x-4)^2+y^2]$$

$$x^2+8x+16+y^2 = 100-20\sqrt{(x-4)^2+y^2} + x^2 - 8x + 16 + y^2$$

Isolate the radical. You will see that the x^2's, y^2's and 16's will cancel and you will add 8*x* to both sides.

$$16x-100 = -20\sqrt{(x-4)^2+y^2}$$

To make this a little easier we are going to divide both sides of the equation by 4 to reduce the large numbers.

$$4x-25 = -5\sqrt{(x-4)^2+y^2}$$

Again we have to square both sides to eliminate the radical. Also make sure to expand the $(x-4)^2$ under the radical.

$$(4x-25)^2 = (-5\sqrt{(x-4)^2+y^2})^2$$

$$(4x-25)(4x-25) = 25(x^2-8x+16+y^2)$$

$$16x^2-100x-100x+625 = 25x^2-200x+400+25y^2$$

$$16x^2-200x+625 = 25x^2-200x+400+25y^2$$

Move all variables to one side of the equation and all constant terms to the other side of the equation.

$$225 = 9x^2+25y^2$$

Divide everything by 225 so the equation is set equal to 1.

$$1=\frac{x^2}{25}+\frac{y^2}{9}$$ The center of this ellipse is located at (0, 0).

It is also possible to write the standard equation of an ellipse by completing the square.

Example #2: Write $25x^2+9y^2+100x+18y=116$ in standard form

Rewrite the terms so the like variables are side by side.

$$25x^2+100x+9y^2+18y=116$$

Remember that before we can complete the square, our quadratic terms must be one. Therefore you must factor a 25 out of the *x* terms and a 9 out of the *y* terms.

$$25(x^2+4x+\_\_\_)+9(y^2+2y+\_\_\_)=116+\_\_\_+\_\_\_$$

$$25(x^2+4x+4)+9(y^2+2y+1)=116+100+9$$

Remember, if you have factored a number out of the terms to complete the square, you must put them back in when adding it to the other side. So the 100 on the right is from 25 times 4 and the 9 is from 9 times 1.

$$25(x+2)^2+9(y+1)^2=225$$

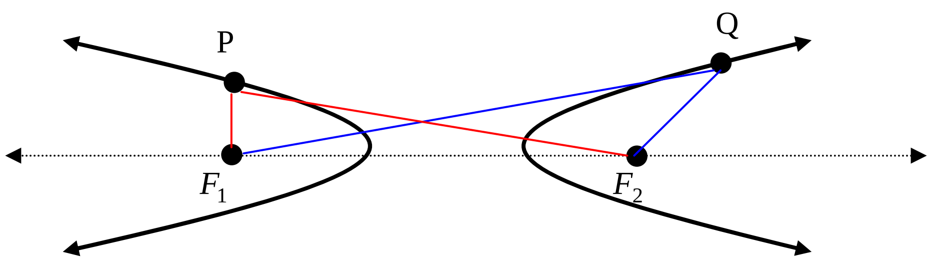
Divide all terms by 225 so the equation is equal to 1.

$$\frac{(x+2)^2}{9}+\frac{(y+1)^2}{25}=1$$

The equation is now in standard from.

Hyperbola

A hyperbola is the set of points P(x, y) in a plane such that the absolute value of the difference between the distances from P to two fixed points F_1 and F_2 called foci, is a constant.



$$|PF_1 - PF_2| = |QF_1 - QF_2|$$

The standard form of the equation of a hyperbola is given below.

horizontal transverse	vertical transverse
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$
(opens left and right like the figure above because the x variable is first)	(opens up and down because the y variable is first)
$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)}{b^2} = 1$	$\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

The center of each hyperbola above is located at (h, k)

Example #1: We can use the definition of hyperbola and the distance formula to find an equation for the hyperbola that contains P(x, y) and has foci at F_1(–5, 0) and F_2(5, 0) and has a constant difference of 6.

distance between (x, y) and (–5, 0) − distance between (x, y) and (5, 0) = 6

$$\sqrt{(x+5)^2 + (y-0)^2} - \sqrt{(x-5)^2 + (y-0)^2} = 6$$

-add $\sqrt{\;}$ to both sides

$\sqrt{(x+5)^2+y^2}=6+\sqrt{(x-5)^2-y^2}$ -square both sides

$(x+5)^2+y^2=36+12\sqrt{(x-5)^2+y^2}+(x-5)^2+y^2$ -expand quantities

$x^2+10x+25+y^2=36+12\sqrt{(x-5)^2+y^2}+x^2-10x+25+y^2$ - isolate the $\sqrt{\ }$

$20x-36=12\sqrt{x^2-10x+25+y^2}$ -divide both sides by 4

$5x-9=3\sqrt{x^2-10x+25+y^2}$ -square both sides

$(5x-9)(5x-9)=(3\sqrt{x^2-10x+25+y^2})^2$

$25x^2-90x+81=9(x^2-10x+25+y^2)$ -multiply quantity by 9

$25x^2-90x+81=9x^2-90x+225+9y^2$ -put in standard form

$16x^2-9y^2=144$ -divide both sides by 144

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

The equation of a hyperbola can be put into standard form the same way we put the equation of an ellipse in standard form except we are subtracting. You need to remember how to complete the square.

Step #1: factor any coefficients that are not = 1 from the quadratic term and the linear term

Step #2: take half of the linear term and square it

Step #3: multiply the answer to step 2 by any number that was factored out and add this to the other side

Example #2: Put this equation in standard form

$4y^2-36x^2-72x+8y=176$ -rearrange terms so like variables are together

$4y^2+8y-36x^2-72x=176$

$$4(y^2+2y+\_\_\_)-36(x^2+2x+\_\_\_)=176+\_\_\_+\_\_\_ \qquad (\textit{step \#1})$$

$$4(y^2+2y+1)-36(x^2+2x+1)=176+4+(-36) \qquad (\textit{step \#2} \text{ and } \textit{step \#3})$$

$$4(y+1)^2-36(x+1)^2=144$$

$$\frac{(y+1)^2}{36}-\frac{(x+1)^2}{4}=1$$

This hyperbola opens up and down and has a center at (–1, –1).

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