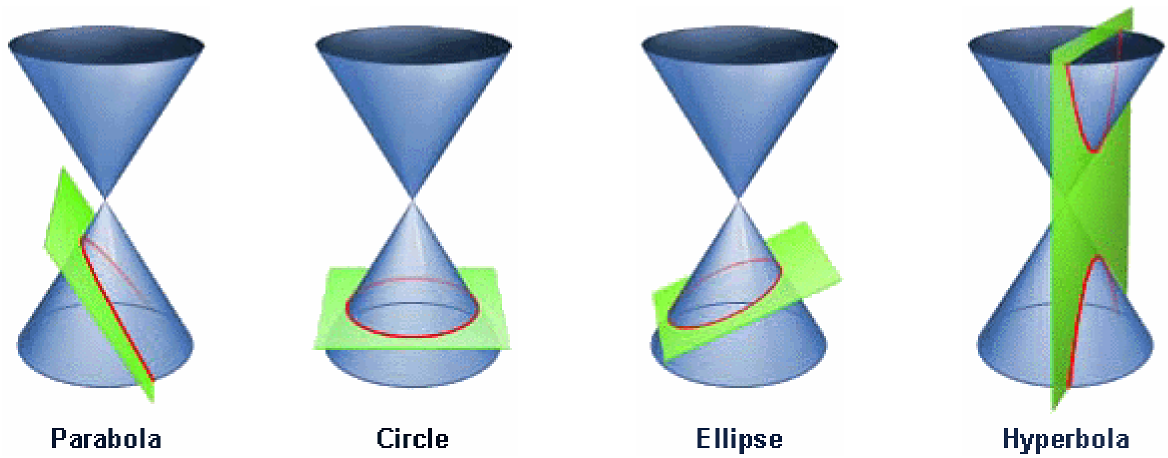
CONIC SECTIONS

In this unit you will be introduced to the general concept of conic sections. We will discuss the particular conic sections of a parabola and a circle.

Conic Sections

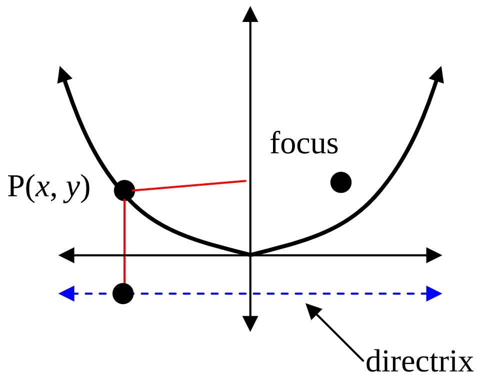
A **conic section** is a type of curve formed by the intersection of a plane and a double-napped cone. There are four types of conic sections; parabola, circle, ellipse, and hyperbola, which are illustrated below. In this unit we will be focusing on the parabola and circle only.

(Illustrations from *http://britton.disted.camosun.bc.ca/jbconics.htm*)

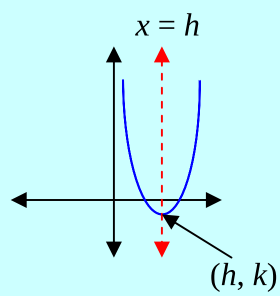
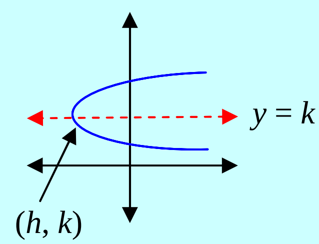
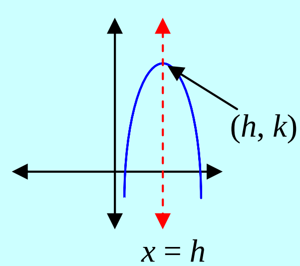
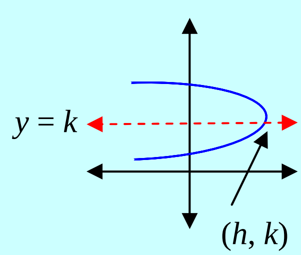


Parabola

A **parabola** is a set of points P(*x*, *y*) in the plane whose distance to a fixed point, called the focus, equals its distance to a point on a fixed line, called the directrix.



The standard equation of a parabola, also known as the **vertex form**, can be derived from this definition and will be done so in the next section. Below is an outline of equations of a parabola.

Equation	$y = a(x-h)^2 + k$	$x = a(y-k)^2 + h$
Vertex	(h, k)	(h, k)
Axis of symmetry	$x = h$	$y = k$
Direction of opening	Up if $a > 0$ Down if $a < 0$	Right if $a > 0$ Left if $a < 0$
Graphs	$\boldsymbol{a > 0}$ 	$\boldsymbol{a > 0}$ 
	$\boldsymbol{a < 0}$ 	$\boldsymbol{a < 0}$ 

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) on a coordinate plane can be found using the following distance formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Example #1: Find the distance between A(2, –3) and B(4, 6)

$$d = \sqrt{(2-4)^2 + (-3-6)^2}$$

$$d = \sqrt{(-2)^2 + (-9)^2}$$

$$d = \sqrt{4+81} = \sqrt{85}$$

This formula can be used to find the equation of a parabola with a given focus and a given directrix.

Example #2: Find the equation of the parabola who has focus (5, 3) and directrix $y = -1$.

a.) the definition of a parabola tells us that the distance between some point (x, y) and the focus (5, 3) is equal to the distance between the same point (x, y) and the directrix $y = -1$ (the point on the directrix will be $(x, -1)$).

b.) set up an equation using the distance formula

distance between $(x,\ y)$ and $(5, 3)$ = distance between (x, y) and $(x, -1)$

$$\sqrt{(x-5)^2 + (y-3)^2} = \sqrt{(x-x)^2 + (y-(-1))^2}$$

$$\sqrt{(x-5)^2 + (y-3)^2} = \sqrt{0^2 + (y+1)^2}$$ square both sides

$$(x-5)^2 + (y-3)^2 = (y+1)^2$$ expand the "y" quantities

$$(x-5)^2 + y^2 - 6y + 9 = y^2 + 2y + 1$$ solve for y

$$(x-5)^2 + 8 = 8y$$ divide everything by 8

$$\frac{1}{8}(x-5)^2 + 1 = y$$

The equation of a parabola with focus (5, 3) and directrix $y = -1$ is $\frac{1}{8}(x-5)^2 + 1 = y$.

Completing the Square

When given an equation of a parabola that is not in standard (vertex) form, it is necessary to put it in that form using a process called **completing the square.** This is a process by which you force a quadratic expression or equation to factor.

1.) Only work with the quadratic term (ax^2) and the linear term (bx) of the standard expression $ax^2 + bx + c$.

2.) Find $\frac{1}{2}$ the linear term (bx), square it $\left(\frac{bx}{2}\right)^2$, add this to the expression.

Example #3: Complete the square for $x^2 + 10x + 0$ to form a perfect square trinomial.

$$x^2 + 10x + 0$$

$$x^2 + 10x + \underline{\ 25\ } + 0$$

$$\frac{1}{2}(10) = 5 \Rightarrow 5^2 = 25$$

$$x^2 + 10x + 25 \Rightarrow (x+5)^2$$

When working with an equation as opposed to an expression like above, you must keep it equal, meaning that when you add something to one side, you must do so to the other side. Follow the example below in which you complete the square to produce the vertex form of a parabola.

Example #4: Write $y^2 - 6y + 39 = 6x$ in vertex form.

Since this equation has only one squared term, we know it is a parabola. We want to complete the square using the terms associated with the variable *y* which means we will be solving the equation for *x* and it will be in the form $x = a(y-k)^2 + h$.

a.) Find half the linear term (6) and square it. This value will be added to the quantity ($y^2 - 6y$), but, to keep the equation equal, you will also have to add this value to the other side.

$$y^2 - 6y + 39 = 6x$$

$$y^2 - 6y + \underline{\quad 9 \quad} + 39 = 6x + \underline{\quad 9 \quad}$$

$$\frac{1}{2}(6) = 3 \Rightarrow 3^2 = 9$$

b.) Our new equation has produced a perfect squared trinomial, $y^2 - 6y + 9$, which can be factored into $(y-3)^2$.

$$(y-3)^2 + 39 = 6x + 9$$

c.) Since we want to solve for *x*, subtract 9 from both sides

$$(y-3)^2 + 30 = 6x$$

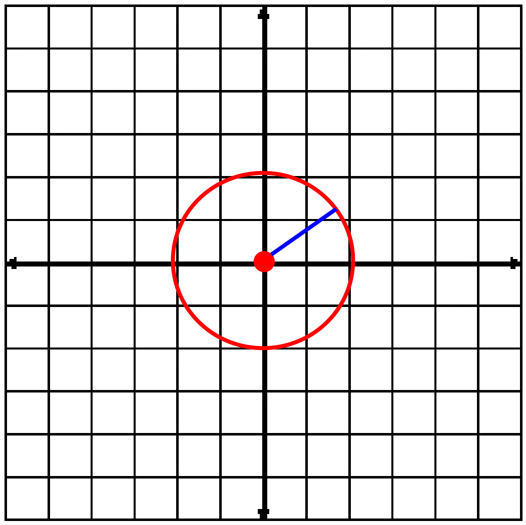
d.) Divide everything by 6 to solve for *x*.

$$\frac{1}{6}(y-3)^2 + 5 = x$$

The vertex form of $y^2 - 6y + 39 = 6x$ is $\frac{1}{6}(y-3)^2 + 5 = x$. The vertex is located at (5, 3), the axis of symmetry is $y = 3$, and the direction of opening is right.

Circles

A **circle** is a set of points in a plane that are equidistant from a given point, called the **center**. Any segment whose endpoints are the center and a point on the circle is a **radius** of the circle.



The standard equation of a circle with center (h, k) and radius r units:

$$(x-h)^2+(y-k)^2=r^2$$

To write an equation of a circle given the center (h, k) and the radius (r):

1.) replace *h* and *k* with the given center.

2.) replace *r* with the given radius

Example #1: Write an equation of a circle whose center is (–1, –5) and whose radius is 2 units.

1.) replace *h*, *k*, and *r* with the given information

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-1))^2+(y-(-5))^2=2^2$$

2.) simplify

$$(x+1)^2+(y+5)^2=4$$

$(x+1)^2+(y+5)^2=4$ is the standard equation of a circle whose center is at (–1, –5) and has a radius of 2 units.

Sometimes it is necessary to complete the square on both variables *x* and *y* to change the equation into standard form. This is done the same way as for parabolas only you will perform the completing the square twice.

Example #2: Find the center and radius of the circle whose equation is
$x^2+14x+y^2+6y+50=0$

1.) subtract 50 from both sides so you are only working with the variable terms

$$x^2+14x+y^2+6y=-50$$

2.) complete the square on *x* and *y*

$$x^2+14x+\_\_\_\_+y^2+6y+\_\_\_\_=-50+\_\_\_\_+\_\_\_\_$$

$$\frac{1}{2}(14)=7\Rightarrow 7^2=49 \qquad \frac{1}{2}(6)=3\Rightarrow 3^3=9$$

3.) remember that whatever you add to the left, you must add to the right also

$$x^2+14x+49+y^2+6y+9=-50+49+9$$

4.) factor the two trinomials on the left and simplify the right

$$(x+7)^2+(y+3)^2=8$$

The center of the circle is located at (–7, –3) and the radius is $\sqrt{8}$ units.

Circle–Line Graph Paper