INVERSES OF MATRICES

In this unit you will explore inverse matrices and use them to solve systems of linear equations.

> During World War II, Navaho code talkers, 29 members of the Navaho Nation, developed a code that was used by the United States Armed Forces.

Matrices

Solving Systems with Matrix Equations

Matrices

Square Matrices

A matrix can be used to encode a message; and another matrix, **it's inverse**, is used to decode a message once it is received.

A **square matrix** is a matrix that has the same number of rows and columns.

Example #1: $\begin{bmatrix} 2 & 3 \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$ $\begin{bmatrix} 4 & -9 & 5 \\ 7 & -3 & 8 \\ 0 & 2 & -1 \end{bmatrix}$ $\begin{bmatrix} -2 & -8 & 4 & 8 \\ 0 & 1 & -3 & 6 \\ 4 & -4 & 0 & 9 \\ 11 & 32 & 57 & 3 \end{bmatrix}$

2 x 2 3 x 3 4 x 4

Identity of Matrices

The Identity Matrix for Multiplication

Let A be a square matrix with n rows and n columns. Let I be a matrix with the same dimensions and with 1's on the main diagonal and 0's elsewhere. Then $AI = IA = A$.

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of a real number and 1 is the same number. The product of a square matrix *B*, and it's identity ***I***, is the matrix *B*.

Example #1: $\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-4 \cdot 1) + (4 \cdot 0) & (-4 \cdot 0) + (4 \cdot 1) \\ (2 \cdot 1) + (8 \cdot 0) & (2 \cdot 0) + (8 \cdot 1) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}$

B x ***I*** = ***B***

Inverse Matrices

The Inverse of a Matrix

Let *A* be a square matrix with *n* rows and *n* columns. If there is an *n* x *n* matrix ***B*** such that *AB = I and BA = I*, then *A* and *B* are inverses of one another. The inverse of matrix ***A*** is denoted by A^{-1}. (Note: $A^{-1} \neq \frac{1}{A}$)

The product of a real number and its multiplicative inverse is 1. The product of a square matrix and its inverse is the identity matrix ***I***.

Example #2: Let C = $\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ and D = $\begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$

To show that C and D are inverses of one another, multiply CD and DC. If the result is the identity matrix for both, they are inverses of each other.

$$CD = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \cdot -7) + (5 \cdot 3) & (2 \cdot 5) + (5 \cdot -2) \\ (3 \cdot -7) + (7 \cdot 3) & (3 \cdot 5) + (7 \cdot -2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$DC = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} (-7 \cdot 2) + (5 \cdot 3) & (-7 \cdot 5) + (5 \cdot 7) \\ (3 \cdot 2) + (-2 \cdot 3) & (3 \cdot 5) + (-2 \cdot 7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determinant of a 2 x 2 Matrix

Each square matrix can be assigned a real number called the ***determinant of the matrix.***

Determinant of a 2 x 2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A, denoted by det(A) or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Matrix A has an inverse, if and only if, $\det(A) \neq 0$.

To use the determinant to find the inverse:

1.) find the difference of the cross products
2.) put this number under 1 and multiply it with the matrix using the following changes
   a.) change the location of a and d in the matrix
   b.) change the signs of b and c in the matrix

Example #3: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If $B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, then $B^{-1} = \dfrac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$

$$= \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

Value of a 3 x 3 Determinant

One way to find the value of a 3 x 3 determinant is called "**expansion of minors**". The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

Example #4: For the determinant $\begin{vmatrix} 4 & 3 & 5 \\ 2 & -1 & -6 \\ 1 & 0 & 8 \end{vmatrix}$,

the minor of 3 is $\begin{vmatrix} \cancel{4} & 3 & \cancel{5} \\ 2 & \cancel{-1} & -6 \\ 1 & \cancel{0} & 8 \end{vmatrix}$ or $\begin{vmatrix} 2 & -6 \\ 1 & 8 \end{vmatrix}$.

the minor of 4 is $\begin{vmatrix} 4 & \cancel{3} & \cancel{5} \\ \cancel{2} & -1 & -6 \\ \cancel{1} & 0 & 8 \end{vmatrix}$ or $\begin{vmatrix} -1 & -6 \\ 0 & 8 \end{vmatrix}$.

To use the expansion of minors with a 3 x 3 determinant, multiply each member of one row by its minor. The signs of the products alternate (+ – +). *Any row of the determinant can be used but using the first row is usually easiest.

Example #5: Evaluate $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$ using the expansion of minors.

$$\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix} \qquad -1\begin{vmatrix} -2 & -5 \\ -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix} \qquad 4\begin{vmatrix} 3 & -5 \\ -3 & 2 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix} \qquad 0\begin{vmatrix} 3 & -2 \\ -3 & -1 \end{vmatrix}$$

$$-1\begin{vmatrix} -2 & -5 \\ -1 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -5 \\ -3 & 2 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ -3 & -1 \end{vmatrix}$$

$-1(-4 - 5) - 4(6 - 15) + 0(-3 - 6)$

$-1(-9) - 4(-9) + 0(-9)$

$9 + 36 + 0 = 45$

The value of the determinant is 45.

Solving Systems with Matrix Equations

2 x 2 systems

A system of linear equations can be written as a matrix equation.

Example #1: $\quad 2a + 4b = -3$
$\qquad\qquad\qquad\quad a - b = 9$

coefficient matrix, ***A***	variable matrix, ***X***	constant matrix, ***B***
$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$	$X = \begin{bmatrix} a \\ b \end{bmatrix}$	$B = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

To solve the system using matrices:

1.) find the inverse of the **coefficient matrix**

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}, \; A^{-1} = \frac{1}{-2-4}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$$

2.) multiply both sides of the equation by A^{-1} (the inverse of the coefficient matrix)

$$\overset{A^{-1}}{-\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} a \\ b \end{bmatrix}} = \overset{A^{-1}}{-\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} -3 \\ 9 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ \frac{-7}{2} \end{bmatrix}$$

$$a = \frac{11}{2} \text{ and } b = \frac{-7}{2}$$

3 x 3 Systems

Matrices can also be used to solve 3 x 3 systems of equations. The process is the same as solving 2 x 2 systems.

1.) Write the system using matrices

$$3x - 2y + z = 0$$
$$2x + 3y = 12$$
$$y + 4z = -18$$

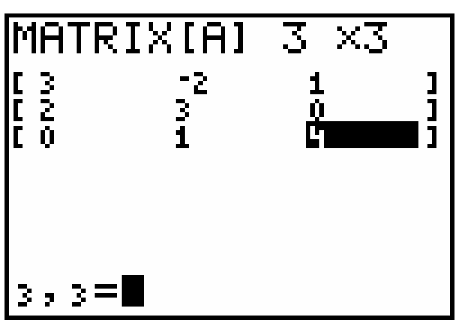
$$\begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ -18 \end{bmatrix}$$

Coefficient matrix   Variable matrix   Constant matrix

2.) Find the inverse of the coefficient matrix. *This will be done using your calculator.

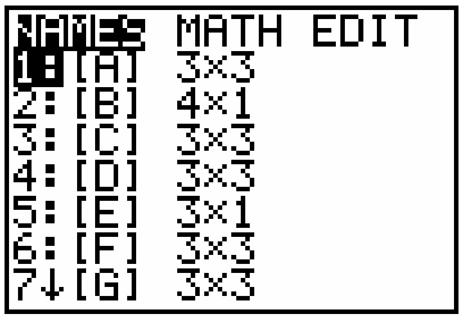
a.) enter the 3 x 3 coefficient matrix into matrix A

$\boxed{2nd}$ $\boxed{x^{-1}}$, move the cursor over to edit and press $\boxed{\text{ENTER}}$, change the dimensions to 3 x 3 if not already set. Proceed to enter the elements of the coefficient matrix, press $\boxed{\text{ENTER}}$ after each entry.

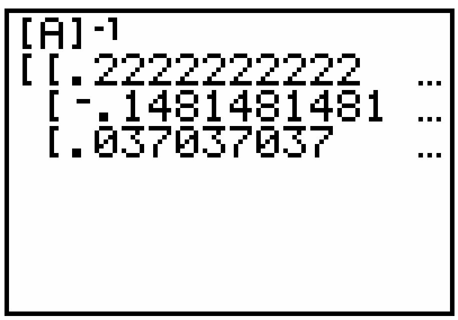


b.) press $\boxed{2\textit{nd}}$ $\boxed{\text{MODE}}$ to return to the main viewing screen.

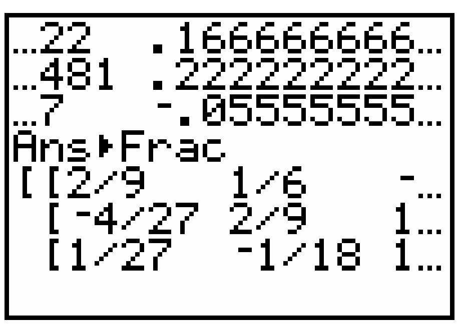
c.) press $\boxed{2\textit{nd}}$ $\boxed{x^{-1}}$ to activate your matrices and choose [A] by pressing $\boxed{\text{ENTER}}$



Your calculator screen should have [A] on it. Now press $\boxed{x^{-1}}$ $\boxed{\text{ENTER}}$



To view the elements in the second and third columns all you have to do is move your cursor to the right. Sometimes it is necessary to write out the inverse of the 3 x 3 and this will be easier if the elements are in fraction form. Press $\boxed{\text{MATH}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$



The inverse of the coefficient matrix is $\begin{bmatrix} \frac{2}{9} & \frac{1}{6} & \frac{-1}{18} \\ \frac{-4}{27} & \frac{2}{9} & \frac{1}{27} \\ \frac{1}{27} & \frac{-1}{18} & \frac{13}{54} \end{bmatrix}$

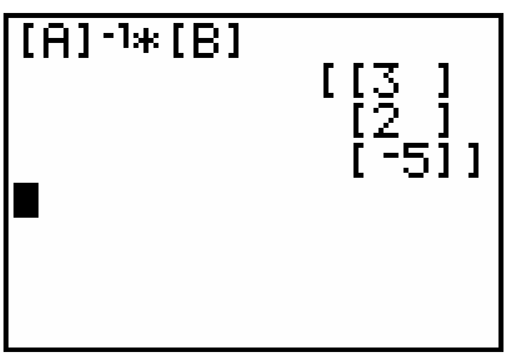
3.) Multiply both sides of the equation by the inverse of the coefficient matrix. This can be done in your calculator once you have it all set up.

$$\begin{bmatrix} \frac{2}{9} & \frac{1}{6} & \frac{-1}{18} \\ \frac{-4}{27} & \frac{2}{9} & \frac{1}{27} \\ \frac{1}{27} & \frac{-1}{18} & \frac{13}{54} \end{bmatrix} \text{x} \begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 4 \end{bmatrix} \text{x} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{6} & \frac{-1}{18} \\ \frac{-4}{27} & \frac{2}{9} & \frac{1}{27} \\ \frac{1}{27} & \frac{-1}{18} & \frac{13}{54} \end{bmatrix} \text{x} \begin{bmatrix} 0 \\ 12 \\ -18 \end{bmatrix}$$

Remember that when you multiply a matrix by its inverse, the result is the identity matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{6} & \frac{-1}{18} \\ \frac{-4}{27} & \frac{2}{9} & \frac{1}{27} \\ \frac{1}{27} & \frac{-1}{18} & \frac{13}{54} \end{bmatrix} \text{x} \begin{bmatrix} 0 \\ 12 \\ -18 \end{bmatrix}$$

a.) enter the constant matrix into your calculator under matrix [B], then multiply $[A]^{-1}[B]$ to produce the result.



The solution to the system is (3, 2, –5).