MATRICES

In this unit you will learn about matrices and how to perform operations on matrices. Matrices are used to organize and store data.

Adding and Subtracting Matrices

Multiplication

Adding and Subtracting Matrices

Types of Matrices

A **matrix** is a rectangular array of numbers enclosed in a single set of brackets. The **dimensions** of a matrix are the number of horizontal rows and the number of vertical columns.

Example #1: $\qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

The matrix A above has the dimensions 2 x 3, 2 rows and 3 columns.

If a matrix has the same number of rows and columns it is called a **square matrix.** A matrix with only one row is called a **row matrix**, and a matrix with only one column is a **column matrix**.

Example #2:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$$

square	row	column
matrix	matrix	matrix

The small letters inside each set of brackets represent the elements of each matrix. The sub numbers that are shown represent the placement of each element. For example a_{23} would represent that "*a*" is located in row 2 column 3.

Matrix Equality

Two matrices are considered equal if they have the same dimensions and the elements placed in corresponding positions are equal. This property can be used to find unknown values with matrices.

Example #3: $\quad \begin{bmatrix} x & 2y \\ 3 & z \\ -2 & 2 \end{bmatrix}$ and $\begin{bmatrix} -3 & 4 \\ 3 & z \\ -2 & \dfrac{6}{3} \end{bmatrix}$ are equal only if $x = -3$ and $y = 2$.

Example #4: Find the values of x and y if $\begin{bmatrix} 3x & z \\ 5 & 7y \end{bmatrix} = \begin{bmatrix} 15 & z \\ 5 & 42 \end{bmatrix}$

    a.) since these two matrices have the same dimensions, we can set the corresponding parts equal to each other and solve.

$$3x = 15 \qquad z = z \qquad 5 = 5 \qquad 7y = 42$$

$$x = 5 \qquad\qquad\qquad\qquad\qquad\qquad y = 6$$

Adding Matrices

The sum of two matrices, of the same size, is formed by adding corresponding elements of each matrix.

Example #5: Add $\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -5 & -7 \\ 4 & 3 \\ 9 & -13 \end{bmatrix}$

We can add the corresponding elements of these matrices because they have the same dimensions.

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -5 & -7 \\ 4 & 3 \\ 9 & -13 \end{bmatrix} = \begin{bmatrix} 3+(-5) & 2+(-7) \\ -1+4 & 0+3 \\ 4+9 & 5+(-13) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 \\ 3 & 3 \\ 13 & -8 \end{bmatrix}$$

Subtracting Matrices

Subtracting matrices is performed in the same manner as adding matrices, and again is only possible if the two matrices have the same dimensions.

Example #6: Subtract $\begin{bmatrix} 5 & -1 & 3 \\ -4 & 0 & -6 \end{bmatrix} - \begin{bmatrix} -2 & 4 & -9 \\ 0 & 4 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 5-(-2) & -1-4 & 3-(-9) \\ -4-0 & 0-4 & -6-(-5) \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 & 12 \\ -4 & -4 & -1 \end{bmatrix}$$

Multiplication

Scalar Multiplication

Recall scalar multiplication of vectors. It was defined that *k*, a real number, was called a scalar. The same applies to scalar multiplication of matrices.

To perform scalar multiplication on matrices means to multiply each element of the matrix by the scalar number given.

Example #1: If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & -2 \end{bmatrix}$, compute $-2A$.

$$-2\begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} (-2)(3) & (-2)(2) & (-2)(-1) \\ (-2)(0) & (-2)(4) & (-2)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -4 & 2 \\ 0 & -8 & 4 \end{bmatrix}$$

Scalar multiplication can be combined with addition or subtractions as shown in the example below.

Example #2: Compute $2\begin{bmatrix} 4 & -2 & 5 \\ 7 & 1 & 8 \end{bmatrix} - 4\begin{bmatrix} -6 & -1 & 0 \\ 1 & 5 & 1 \end{bmatrix}$

$$\begin{bmatrix} (2)(4) & (2)(-2) & (2)(5) \\ (2)(7) & (2)(1) & (2)(8) \end{bmatrix} + \begin{bmatrix} (-4)(-6) & (-4)(-1) & (-4)(0) \\ (-4)(1) & (-4)(5) & (-4)(1) \end{bmatrix}$$

*Notice that (-4) was used to multiply the second matrix and the sign between the two matrices was changed to (+). This is suggested so that you don't get confused about the signs.

$$\begin{bmatrix} 8 & -4 & 10 \\ 14 & 2 & 16 \end{bmatrix} + \begin{bmatrix} 24 & 4 & 0 \\ -4 & -20 & -4 \end{bmatrix} = \begin{bmatrix} 32 & 0 & 10 \\ 10 & -18 & 12 \end{bmatrix}$$

Matrix Multiplication

It is possible to multiply two matrices providing the number of columns in the first matrix matches the number of rows in the second matrix. Matrix multiplication involves multiplication and addition.

Example #3: If matrix A is a 2 x 3 matrix and matrix B is a 3 x 2 matrix, it is possible to multiply the two together and produce a product matrix that has dimensions 2 x 2.

$$A \begin{bmatrix} 4 & -2 & 3 \\ 0 & 3 & -4 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}$$

$$A_{2\times 3} \qquad\qquad B_{3\times 2}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\text{Columns} \qquad\qquad \text{Rows}$$

Let's take a closer look at how this works. Since the number columns in matrix A match the number of rows in matrix B, these two matrices can be multiplied. Notice the numbers remaining, 2 and 2, this will represent the dimensions of the product, 2 x 2.

To multiply the two matrices together you will begin by:

a.) multiplying each element in the first row with each element of the first column as indicated with the blue print. The sum of these products will result in the first element in the first row first column of the product.

$$\begin{bmatrix} 4 & -2 & 3 \\ 0 & 3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} (4)(-1)+(-2)(3)+(3)(0) & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

b.) You will repeat this process to produce the second element of the first row using the first row of the first matrix and the second column of the second matrix (red print).

$$\begin{bmatrix} 4 & -2 & 3 \\ 0 & 3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -10 & (4)(2)+(-2)(-2)+(3)(5) \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

c.) To produce the second row of the product, you will repeat the process from (a) and (b) above using the second row of the first matrix.

$$\begin{bmatrix} 4 & -2 & 3 \\ 0 & 3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -10 & 27 \\ 9 & \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 3 \\ 0 & 3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -10 & 27 \\ 9 & -26 \end{bmatrix}$$

$$AB = \begin{bmatrix} -10 & 27 \\ 9 & -26 \end{bmatrix}$$

It is important to note that matrix multiplication is not commutative; therefore, AB ≠ BA.

At this point we will look at how matrices are used in application situations.

Example #4: A high school basketball team has played four games this season. Matrix A shows the number of field goal shots, both 3 pointers and 2 pointers, and the number of free throws (foul shots) scored in each game. Matrix F represents the point value for each shot. We will multiply these matrices together to find the number of points scored in each game.

$$\begin{matrix} & \begin{matrix} \text{3 point} & \text{2 point} & \text{foul} \\ \text{shots} & \text{shots} & \text{shots} \end{matrix} \\ \begin{matrix} \text{Game 1} \\ \text{Game 2} \\ \text{Game 3} \\ \text{Game 4} \end{matrix} & \begin{bmatrix} 5 & 12 & 7 \\ 4 & 20 & 8 \\ 3 & 18 & 12 \\ 4 & 5 & 10 \end{bmatrix} = A \end{matrix} \qquad \begin{matrix} \text{3 pointers} \\ \text{field goals} \\ \text{foul shots} \end{matrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = F$$

$$\begin{bmatrix} 5(3) + (12)(2) + (7)(1) \\ 4(3) + 20(2) + 8(1) \\ 3(3) + 18(2) + 12(1) \\ 4(3) + 5(2) + 10(1) \end{bmatrix} = \begin{bmatrix} 46 \\ 60 \\ 57 \\ 32 \end{bmatrix}$$

The team scored 46 points in the first game, 60 points in the second game, 57 in the third game, and 32 points in the fourth game.