TRIGONOMETRY FUNCTIONS OF ANY ANGLE

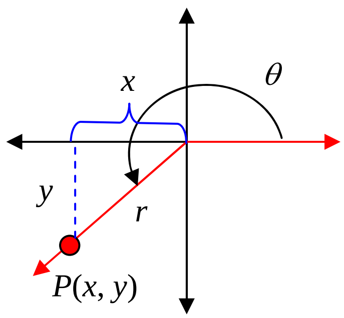
In this unit, additional topics on trigonometry will be presented. You will learn more about angles of rotation and expand your knowledge of trig functions to include finding values for any angle.

General Angle and Real Number Domains

Trigonometric Functions with Angle Domains

If x and y are the coordinates of a point on the terminal side of θ in standard position, you are able to find the values for the trigonometric functions of θ.

Let $P(x, y)$ be a point on the terminal side of θ in standard position. The distance from the origin to P is given by $r = \sqrt{x^2 + y^2}$.



*By drawing a perpendicular from $P(x, y)$ to the x-axis, you construct what is called a **reference triangle**.

$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad$ $\cot\theta = \frac{x}{y}$

Example #1: If *P*(–2, –3) is a point on the terminal side of θ in standard position, find the exact values of the six trig functions of θ.

1.) We know *x*, and *y*, so we need to find *r*

$x = -2$ $y = -3$ $r = ?$

$$r = \sqrt{(-2)^2 + (-3)^2}$$

$$r = \sqrt{4+9}$$

$$r = \sqrt{13}$$

2.) Now we want to use $x = -2$, $y = -3$ and $r = \sqrt{13}$ to substitute into all the trig functions.

*Remember to simplify any radicals in the denominator.

$$\sin\theta = \frac{y}{r} \Rightarrow \frac{-3}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} = \frac{-3\sqrt{13}}{13}$$

$$\csc\theta = \frac{r}{y} \Rightarrow -\frac{\sqrt{13}}{3}$$

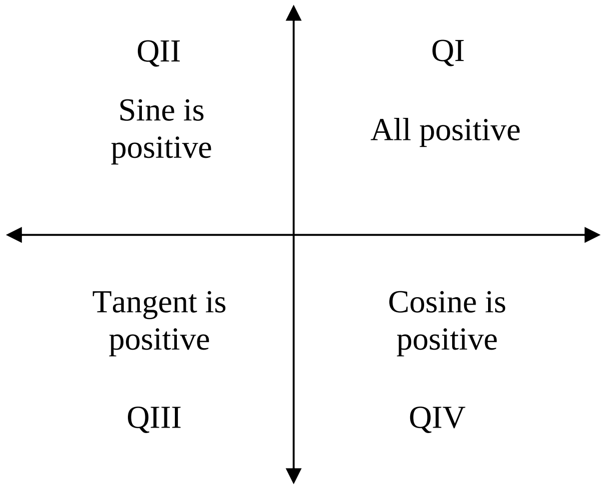
$$\cos\theta = \frac{x}{r} \Rightarrow \frac{-2}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} = \frac{-2\sqrt{13}}{13}$$

$$\sec\theta = \frac{r}{x} \Rightarrow -\frac{\sqrt{13}}{2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \frac{-3}{-2} = \frac{3}{2}$$

$$\cot\theta = \frac{x}{y} \Rightarrow \frac{2}{3}$$

*If given a quadrant that contains the terminal side of θ and the exact value of one trigonometric function, you can find the values of the other trig functions. **Keep in mind that the signs of the trig functions are different in each quadrant.** Study the illustration below.



Example #2: If the terminal side of θ lies in QIV and the $\cos\theta = \frac{5}{13}$; find $\sin\theta$.

1.) We know the $\cos\theta = \frac{x}{r}$ and we are given the $\cos\theta = \frac{5}{13}$ so x = 5 and r = 13.

2.) We need to find y because $\sin\theta = \frac{y}{r}$

$$r = \sqrt{x^2 + y^2}$$

$$13 = \sqrt{5^2 + y^2}$$

$$169 = 25 + y^2$$

$$144 = y^2$$

$$\pm 12 = y$$

3.) We need to determine whether to use +12, or –12 for the sine function. In QIV, y is negative so we need to use the –12.

Therefore, $\sin\theta = \frac{-12}{13}$.

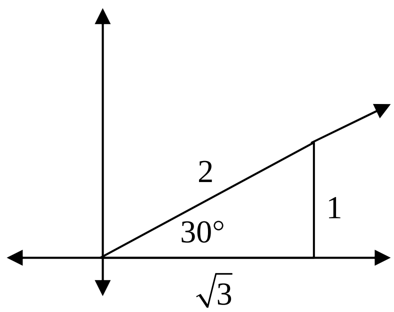
Trig Functions of Any Angle

$30° - 45° - 60°$ Angles

*There are certain angles whose exact trig functions can be found without a calculator. These angles are $30°, 45°, 60°$ and any angle having any of these three as reference angles.

The ratios of the lengths of the sides of each triangle is shown below. Throughout this unit, it will be helpful for you to learn the exact values of sin, cos, and tan of these angles.

$30°$ Triangle

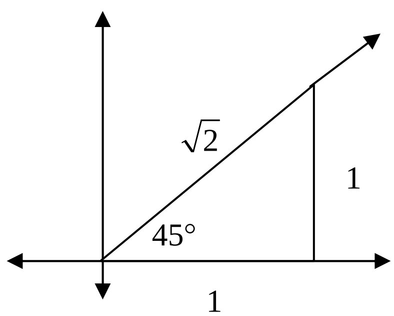


$$\sin 30 = \frac{1}{2}$$

$$\cos 30 = \frac{\sqrt{3}}{2}$$

$$\tan 30 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$45°$ Triangle

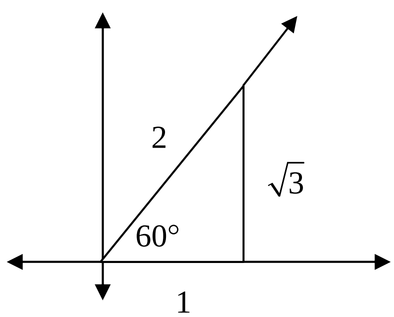


$$\sin 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45 = 1$$

$60°$ Triangle



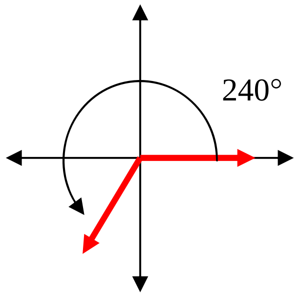
$$\sin 60 = \frac{\sqrt{3}}{2}$$

$$\cos 60 = \frac{1}{2}$$

$$\tan 60 = \sqrt{3}$$

Example #1: Find the exact values of $\sin 240°, \cos 240°, \tan 240°$.

1) Find the reference angle of $240°$.



$$\theta_{ref} = |180 - 240| = 60°$$

2) $240°$ is in QIII where only the tangent function is positive.

3) Using the reference angle of $60°$ the

$$\sin 240 = -\frac{\sqrt{3}}{2}, \quad \cos 240 = -\frac{1}{2}, \quad \tan 240 = \sqrt{3}$$

Locating a Point

To find exact coordinates of a point $P(r\cos\theta, r\sin\theta)$, located at the intersection of a circle with a radius of r and the terminal side of an angle given:

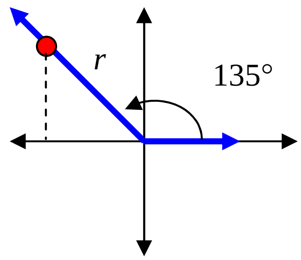
1.) draw a diagram to find the reference angle

2.) find the cos and sin of the reference angle

3.) determine the point $(r\cos\theta, r\sin\theta)$

Example #2: Find the exact location of *P*, with a radius of 7 and the terminal side of θ at $135°$.

1.)



$$\theta_{ref} = |180 - 135| = 45°$$

2.) $\cos 45 = \frac{\sqrt{2}}{2}$ $\rightarrow$ negative in QII, so use $-\frac{\sqrt{2}}{2}$

$\sin 45 = \frac{\sqrt{2}}{2}$

3.) Determine the point $(r\cos\theta, r\sin\theta)$

$$P\left(-\frac{7\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$$