COMPLEX ZEROS AND RATIONAL FUNCTIONS

In this unit you will learn how to find complex zeros of rational functions using the Quadratic Formula. You will also explore the process of finding the domain, vertical asymptotes and horizontal asymptotes of rational functions.

Complex Zeros

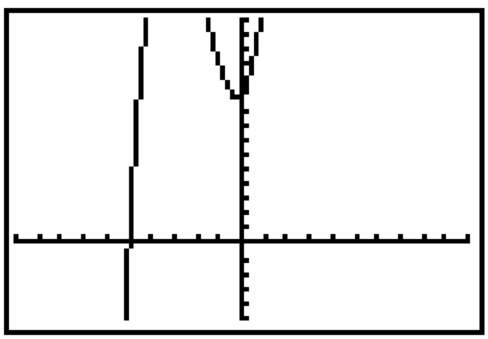
To find any complex zero, we will use the "rational root theorem" that we used for finding rational zeros.

Example #1: Find all zeros of $f(x) = x^3 + 5x^2 + 2x + 10$

1.) factors of 10: $\pm 1, \ \pm 2, \ \pm 5, \ \pm 10$
factors of 1: ± 1

2.) all possible zeros: $\frac{\pm 1}{1}, \ \frac{\pm 2}{1}, \ \frac{\pm 5}{1}, \ \frac{\pm 10}{1}$

3.) Graph $x^3 + 5x^2 + 2x + 10$ on your calculator under Y=. Set the window so that Ymin = –5 and Ymax = 15



4.) Use the graph to determine where there is a zero. In this case it is –5. Use synthetic division to prove this.

–5	1	5	2	10
		–5	0	–10
	1	0	2	0

The quotient $x^2 + 0x + 2$ will now be solved for the remaining zeros.

$$x^2 + 2 = 0$$

$$x^2 = -2$$

$$\sqrt{x^2} = \sqrt{-2}$$

$$x = \pm\sqrt{-2}$$

$$x = \pm i\sqrt{2}$$

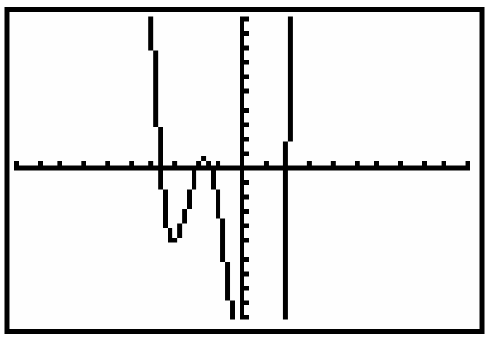
The three roots of $x^3 + 5x^2 + 2x + 10$ are –5, $+i\sqrt{2}$, $-i\sqrt{2}$.

Example #2: Find all zeros for $f(x) = x^4 + 5x^3 + x^2 - 20x - 20$

1.) factors of 20: ± 1, ± 2, ± 4, ± 5, ± 10, ± 20
factors of 1: ± 1

2.) possible zeros: $\frac{\pm 1}{1}$, $\frac{\pm 2}{1}$, $\frac{\pm 4}{1}$, $\frac{\pm 5}{1}$, $\frac{\pm 10}{1}$, $\frac{\pm 20}{1}$

3.) Graph $x^4 + 5x^3 + x^2 - 20x - 20$ on your calculator under Y=



4.) From the graph there appears to be a zero at 2 and –2. Let's use synthetic division with 2 first.

2	1	5	1	–20	–20
		2	14	30	20
	1	7	15	10	0

Since our quotient $x^3 + 7x^2 + 15x + 10$ has a degree larger than 2, repeat the process of synthetic division using the other possible zero (–2).

–2	1	7	15	10
		–2	–10	–10
	1	5	5	0

The new quotient $x^2 + 5x + 5$ can now be solved using the quadratic formula.

The Quadratic Formula

For any quadratic in standard form $ax^2 + bx + c = 0$, the quadratic formula can be used to solve the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$ $b = 5$ $c = 5$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(5)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 20}}{2}$$

$$x = \frac{-5 \pm \sqrt{5}}{2}$$

There are 4 zeros for this function: 2, –2, $\frac{-5+\sqrt{5}}{2}$, and $\frac{-5-\sqrt{5}}{2}$

Rational Functions

Just as rational numbers are defined in terms of quotients of integers, rational functions are defined in terms of quotients of polynomials.

$$f(x) = \frac{x+2}{x^2+5x+6} \qquad g(x) = \frac{7}{x} \qquad h(x) = \frac{x^2-7x+10}{x-2} \qquad f(x) = \frac{x^4-1}{x}$$

Domain

The rational function $f(x) = \frac{1}{x}$ is undefined when $x = 0$.

In general, the domain of a rational function is the set of all real numbers except those that make the denominator equal to 0.

Example #1: Find the domain of $f(x) = \frac{2x^2-2x-4}{x^2-9}$

1.) factor the denominator x^2-9

$(x + 3)(x - 3)$

2.) set each factor of the denominator equal to zero and solve.

$x + 3 = 0 \qquad x - 3 = 0$

$x = -3 \qquad x = 3$

3.) since $x = -3$ and $x = 3$ will result in a denominator of zero, these two values must be excluded from the set of all real numbers.

Therefore, the domain of the function $f(x) = \frac{2x^2-2x-4}{x^2-9}$ will be all real numbers except 3 and –3.

Vertical Asymptotes

Rational functions can have vertical asymptotes at an excluded value.

An ***asymptote*** is a line that a curve approaches (but does not read) as its *x*- or *y*-values become very large or very small.

> **Vertical Asymptotes**
>
> If $x - a$ is a factor of the denominator of a rational function but not a factor of its numerator, then $x = a$ is a vertical asymptote of the graph of the function.

Example #2: Identify all vertical asymptotes of $g(x) = \dfrac{2x}{x^2 - 3x + 2}$

1.) factor the denominator

$$g(x) = \frac{2x}{x^2 - 3x + 2} = \frac{2x}{(x-2)(x-1)}$$

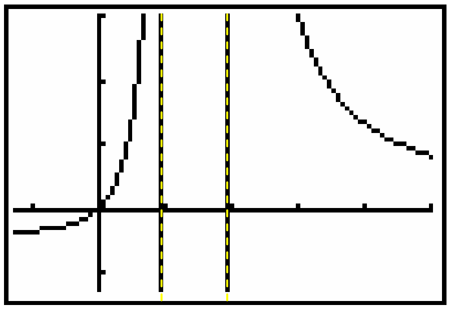
Since neither factor of the denominator is a factor of the numerator, the equations for the vertical asymptotes are $x = 2$ and $x = 1$.

*If a factor of the denominator ***is*** a factor of the numerator, this means there is a hole in the graph at that point. We will discuss this at a later time.

If you graph the functions $g(x) = \dfrac{2x}{x^2 - 3x + 2}$ in Y= on your calculator, you will see the two vertical asymptotes at $x = 1$ and $x = 2$.

Use the window Xmin = –1.25
Xmax = 5

Ymin = –1.25
Ymax = 3



*Note: You can explore the graph further by setting the Ymin to -100 and Ymax to 100. These settings enable you to see what is happening in the graph between the vertical asymptotes.

Example #3: Identify all vertical asymptotes for the function $f(x) = \dfrac{3x+5}{x-2}$

Since the denominator does not need factored and the numerator can not be factored, set the denominator equal to zero and solve for *x*.

$x - 2 = 0$

$x = 2$

The vertical asymptote is $x = 2$.

Horizontal Asymptotes

Horizontal Asymptotes

Let $R(x) = \frac{P}{Q}$ be a rational function, where P and Q are polynomials.

a.) If the degree of P is less than the degree of Q, then $y = 0$ is the equation of the horizontal asymptote.

b.) If the degree of P is equal to the degree of Q and a and b are the leading coefficients of P and Q, then $y = \frac{a}{b}$ is the equation of the horizontal asymptote.

c.) If the degree of P is greater than the degree of Q, then the graph has no horizontal asymptote.

Example #4: Identify all vertical and horizontal asymptotes of $g(x) = \frac{x^3}{x^2 + x - 2}$.

1.) factor the denominator

$$g(x) = \frac{x^3}{x^2 + x - 2} = \frac{x^3}{(x+2)(x-1)}$$

2.) the vertical asymptotes of this function are $x = -2$ and $x = 1$

3.) determine the degree of the numerator and denominator

degree of x^3 is 3
degree of $x^2 + x - 2$ is 2

Since the degree of the numerator is greater than the degree of the denominator, there is **no** horizontal asymptote.

Example #5: Identify all asymptotes of $f(x)=\dfrac{x^2+4x-12}{3x^2-12}$

1.) factor both the numerator and denominator

$$f(x)=\frac{x^2+4x-12}{3x^2-12}=\frac{(x-2)(x+6)}{3(x-2)(x+2)}$$

2.) There is only one vertical asymptote because there is a common factor in the numerator and denominator; we will discuss this later. The vertical asymptote is $x = -2$.

3.) The degree of the numerator is 2 and the degree of the denominator is 2. This means that the horizontal asymptote will be $y=\dfrac{a}{b}=\dfrac{1}{3}$.

Hole in Graph

If $x - b$ is a factor of both the numerator and denominator of a rational function, then there is a **hole in the graph** of the function when $x = b$.

Refer back to the previous example (example #5). Since $(x - 2)$ is a common factor in the numerator and denominator, we can say that the graph has a hole when $x = 2$.

Example #6: Identify all vertical and horizontal asymptotes and any holes in the graph of $h(x)=\dfrac{x^3+3x^2}{x^2+2x-3}$

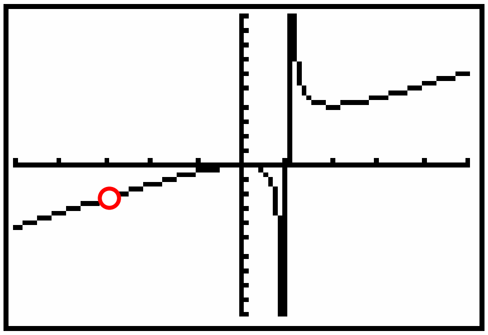
1.) factor

$$h(x)=\frac{x^3+3x^2}{x^2+2x-3}=\frac{x^2(x+3)}{(x+3)(x-1)}$$

vertical asymptote $x = 1$

2.) numerator has a degree of 3 which is greater than the degree of the denominator, so there is no horizontal asymptote

3.) the numerator and denominator share a common factor, so there is a hole in the graph at $x = -3$.



When you graph this on your calculator, you will not see the hole in the graph as shown above; however, if you press TRACE , and then enter -3 for “x =”, you will find that y has no value at $x = -3$.