POLYNOMIAL FUNCTIONS

In this unit you will learn how to find rational zeros and zeros of polynomial functions. You will be introduced to the Remainder Theorem and the Factor Theorem and reintroduced to synthetic division of polynomials. **It is necessary to have at least a TI-83 or TI-83 Plus calculator for this unit. As they become available, upgrades to these models are even better.**

Polynomial Functions

Previously you have studied the following functions:

$f(x) = b$	constant function
$f(x) = ax + b \quad a \neq 0$	linear function
$f(x) = ax^2 + bx + c \quad a \neq 0$	quadratic function
$f(x) = ax^3 + bx^2 + cx + d \quad a \neq 0$	cubic function

Notice the pattern in each equation. The terms in each are in the form ax^n where n is a nonnegative integer and a is a real number. All these functions are special cases of the general class of functions called polynomial functions.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... a_1 x^1 + a_0; \quad a_n \neq 0$$

The function above is called an *nth* degree polynomial function or a polynomial of degree *n*. The numbers $a_n, a_{n-1}, ... a_1, a_0$ are the coefficients.

A nonzero constant function is a zero degree polynomial.

A linear function is a first-degree polynomial.

A quadratic function is a second-degree polynomial.

The coefficients and domain of a polynomial function may be complex numbers, real numbers, rational numbers, or integers depending on the problem.

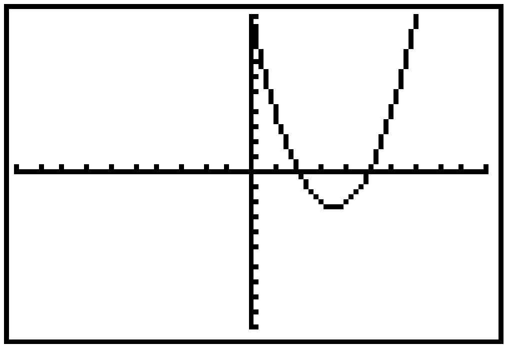
The zero of a polynomial $P(x)$ is the solution or root of the equation $P(x) = 0$ if:

$$P(r) = 0 \qquad (r \text{ is a number})$$

If the coefficients of a polynomial $P(x)$ are real numbers, then the real zero is simply an x-intercept for the graph $y = P(x)$.

Example #1: Find the real zeros of $P(x) = x^2 - 7x + 10$

The graph of *P* is shown below:



The *x*-intercepts 2 and 5 are real zeros of $P(x) = x^2 - 7x + 10$ because *P*(2) and *P*(5) are both equal to 0.

a.) $P(2) = (2)^2 - (7)(2) + 10$

$P(2) = 4 - 14 + 10$

$P(2) = 0$

b.) $P(5) = (5)^2 - 7(5) + 10$

$P(5) = 25 - 35 + 10$

$P(5) = 0$

The *x*-intercepts 2 and 5 are also solutions or roots for the equation $x^2 - 7x + 10 = 0$ when solved by factoring:

$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$x - 5 = 0 \qquad x - 2 = 0$

$x = 5 \qquad x = 2$

Synthetic Division

Another form of division is called synthetic division. Synthetic division can be used to divide a polynomial only by a linear binomial of the form $x - r$ and only uses the coefficients of each term. (When using nonlinear divisors, use long division as discussed in the previous section.)

Example #1: Use synthetic division to find the quotient for the division problem.

$$(x^3 + 3x^2 - 4x - 12) \div (x - 2)$$

Step #1: Write out the coefficients of the polynomial, and then write the *r*-value, 2, of the divisor $x - 2$. **Notice that you use the opposite of the divisor sign**. Write the first coefficient, 1, below the line.

2	1	3	–4	–12
	1			

Step #2: Multiply the *r*-value, 2, by the number below the line, 1, and write the product, 2, below the next coefficient.

2	1	3	–4	–12
		2		
	1			

Step #3: Write the **sum** (not the difference) of 3 and 2, (5), below the line. Then, multiply 2 by the number below the line, 5, and write the product, 10, below the next coefficient.

2	1	3	–4	–12
		2	10	
	1	5		

Step #4: Write the sum of –4 and 10, (6), below the line. Multiply 2 by the number below the line, 6, and write the product, 12, below the next coefficient.

2	1	3	–4	–12
		2	10	12
	1	5	6	0

The remainder is 0, and the resulting numbers 1, 5 and 6 are the coefficients of the quotient. The quotient will start with an exponent that is one less than the dividend.

$$1x^2 + 5x + 6$$

The quotient is $x^2 + 5x + 6$.

Thus, $(x^3 + 3x^2 - 4x - 12) \div (x - 2) = x^2 + 5x + 6$.

Now let's try one that has a remainder.

Example #2: Use synthetic division to find the quotient and remainder for the division problem.

Remember to bring down the coefficient of the first term of the polynomial that is being divided (1 in this case) and continue on by multiplying by the divisor (*r*-value of the linear divisor) and adding.

$$(x^3 - 2x^2 - 22x + 40) \div (x - 4)$$

4	1	–2	–22	40
		4	8	–56
	1	2	–14	–16

Since there is a remainder in this problem, the answer is written using a fraction with the divisor, $x – 4$, as the denominator. Notice that the sign between the last term and the fraction is the same as the sign of the remainder.

$$1x^2 + 2x - 14 - \frac{16}{x - 4}$$

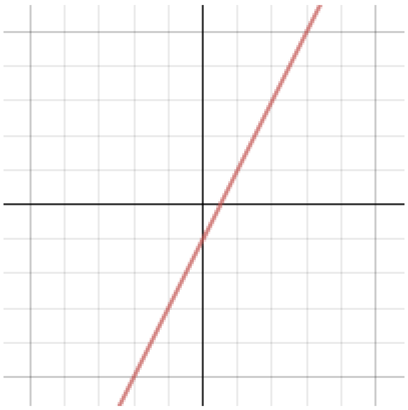
Thus, $(x^3 - 2x^2 - 22x + 40) \div (x - 4) = x^2 + 2x - 14 - \dfrac{16}{x - 4}$.

Division of two polynomials is not always a polynomial as is shown in this example. Therefore polynomials are **not closed** under the operation of division. In the fourth term of the answer, the fraction has $x – 4$ in the denominator. In a polynomial, there **cannot** be any variables in the denominator of a term.

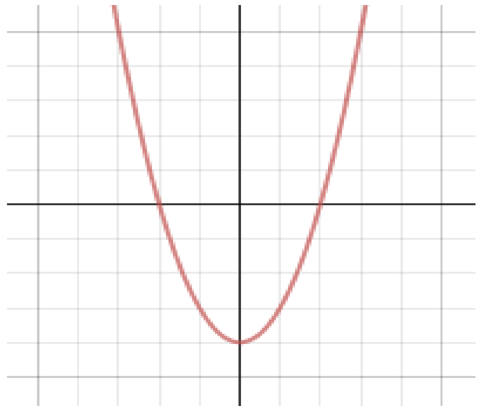
Graphs of Polynomial Functions

The degree of a polynomial function affects the shape of its graph. The graphs below show the general shapes of several polynomial functions. The graphs show the maximum number of times the graph of each type of polynomial may cross the *x*-axis. For example, a polynomial function of degree 4 ***may*** cross the *x*-axis a maximum of 4 times.

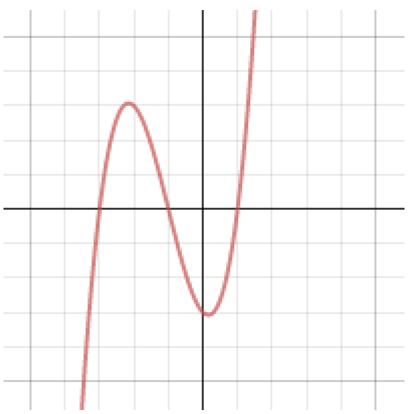
Linear Function
Degree 1



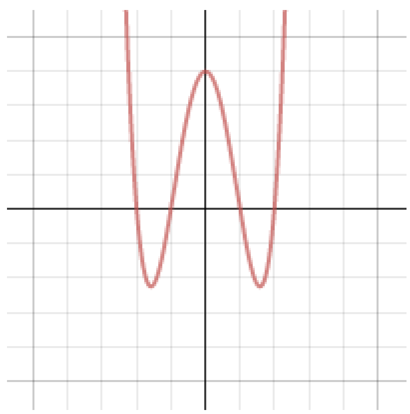
Quadratic Function
Degree 2



Cubic Function
Degree 3



Quartic Function
Degree 4



Quintic Function
Degree 5

Notice the general shapes of the graphs of odd degree polynomial functions and even degree polynomial functions.

- The **degree** and **leading coefficient** of a polynomial function affects the graph's **end behavior**.
- End behavior is the **direction** of the graph to the **far left** and to the **far right**.

The chart below summarizes the end behavior of a Polynomial Function.

Degree	Leading Coefficient	End behavior of graph
Even	**Positive**	Graph goes up to the far left and goes up to the far right.
Even	**Negative**	Graph goes down to the far left and down to the far right.
Odd	**Positive**	Graph goes down to the far left and up to the far right.
Odd	**Negative**	Graph goes up to the far left and down to the far right.

QuickTime Graphics Forms of Polynomial Functions (06:05)

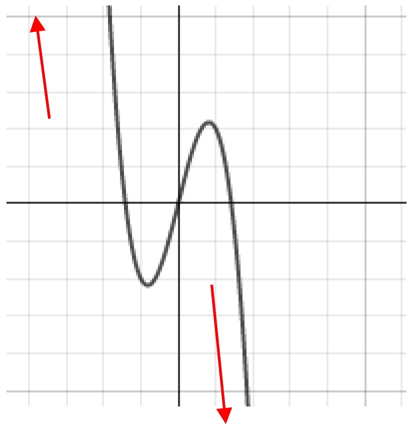
Example #1: Determine the end behavior of the graph of the polynomial function, $y = -2x^3 + 4x$.

The leading term is $-2x^3$.

Since the degree is odd and the coefficient is negative, the end behavior is up to the far left and down to the far right.

Check by using a graphing calculator or click here to navigate to an online grapher.

$y = -2x^3 + 4x$



Odd **Negative** Graph goes up to the far left and down to the far right.

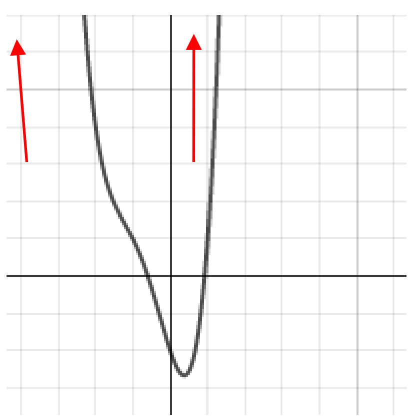
Example #2: Determine the end behavior of the graph of the polynomial function, $y = x^4 + 3x^3 + 2x^2 - 3x - 2$.

The leading term is $1x^4$.

Since the degree is even and the coefficient is positive, the end behavior is up to the far left and up to the far right.

Check by using a graphing calculator or click here to navigate to an online grapher.

$y = x^4 + 3x^3 + 2x^2 - 3x - 2$



Even	**Positive**	Graph goes up to the far left and goes up to the far right.

Example #3: Determine the end behavior of the graph of the polynomial function, $y = -5x + 4 + 2x^3$.

Rearrange the function so that the terms are in descending order.

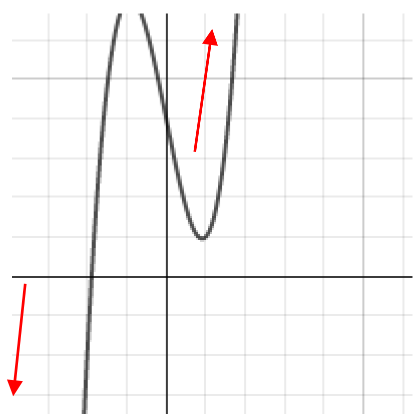
$y = 2x^3 - 5x + 4$

The leading term is $2x^3$.

Since the degree is odd and the coefficient is positive, the end behavior is down to the far left and up to the far right.

Check by using a graphing calculator or click here to navigate to an online grapher.

$y = -5x + 4 + 2x^3$



Odd	**Positive**	Graph goes down to the far left and up to the far right.

Now, let's practice determining the end behavior of the graphs of a polynomial.

Determine the end behavior of the graph of the polynomial function, $y = -2x^4 + 5x^2 - 3$.

What is the degree of the function? $y = -2x^4 + 5x^2 - 3$

Click here" to check your answer.

The degree is 4.

What is the leading term? $y = –2x^4 + 5x^2 – 3$

Click here" to check your answer.

The leading term is $–2x^4$.

What is the end behavior of the function? $y = –2x^4 + 5x^4 – 3$

Click here" to check your answer.

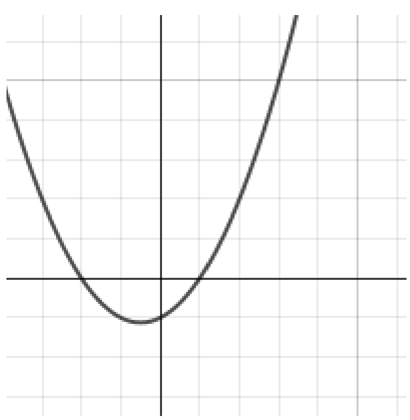
Since the degree is even and the coefficient is negative,
the end behavior is down to the far left and down to the far right.

The real roots or zeros are the *x*-values of the coordinates where the polynomial crosses the *x*-axis.

By examining the graph of a polynomial function, the following can be determined:

- if the graph represents an odd-degree or an even degree polynomial
- if the leading coefficient if positive or negative
- the number of real roots or zeros.

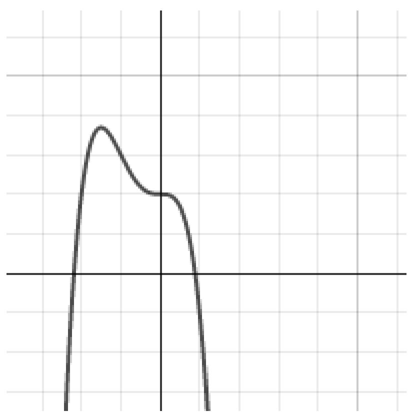
Example #4: For the graph, describe the end behavior, (a) determine if the leading coefficient is positive or negative and if the graph represents an odd or an even degree polynomial, and (b) state the number of real roots (zeros).



(a) The end behavior is up for both the far left and the far right; therefore this graph represents an even degree polynomial and the leading coefficient is positive.

(b) The graph crosses the *x*-axis in two points so the function has two real roots (zeros).

Example #5: For the graph, describe the end behavior, (a) determine if the leading coefficient is positive or negative and if the graph represents an odd or an even degree polynomial, and (b) state the number of real roots.

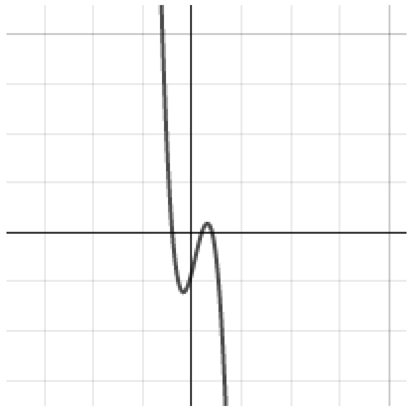


(a) The end behavior is down for both the far left and the far right. Therefore this graph represents an even degree polynomial and the leading coefficient is negative.

(b) The graph crosses the *x*-axis in two points so the function has two real roots (zeros).

Now, let's practice examining a graph and determining the characteristics of its equation.

Examine this graph closely, and then answer the questions that follow about the equation of the graph.



Is the leading coefficient of the equation positive or negative and why?

Click here" to check your answer.

Negative, because it is going up to the far left and down to the far right.

Does the polynomial have an odd or even degree?

Click here" to check your answer.

Odd degree

How can the leading coefficient and the degree of the polynomial be determined by looking at the graph?

Click here" to check your answer.

The end behavior of the graph is down for the far left and up for the far right.

The equation has how many real roots (zeros)?

Click here” to check your answer.

The graph crosses the *x*-axis at three points, thus, the function has three real zeros.

QuickTime Properties of Polynomial Graphs (10:42)

Rational Zeros

Factor Theorem

If r is a zero of the polynomial $P(x)$, then $x - r$ is a factor of $P(x)$. Conversely, if $x - r$ is a factor of $P(x)$, then r is a zero of $P(x)$.

Example #1: Use the Factor Theorem to show $x + 1$ is a factor of $P(x) = x^{23} + 1$

1.) solve $x + 1$ for x by setting it equal to zero

$$x + 1 = 0$$

$$x = -1$$

2.) substitute –1 for x in $P(x) = x^{23} + 1$

$$P(-1) = (-1)^{23} + 1$$

$$P(-1) = -1 + 1$$

$$P(-1) = 0$$

Since the result is 0, this shows that $x + 1$ is a factor of $P(x) = x^{23} + 1$.

Rational Root Theorem

Let P be a polynomial function with integer coefficients in standard form. If $\frac{p}{q}$ (in lowest terms) is a root of $P(x) = 0$ then:

- p is a factor of the constant term of P
- q is a factor of the leading coefficient of P

Example #2: Find all rational roots of $18x^3 + 9x^2 - 23x + 6 = 0$

1.) List factors of the constant (p) 6: $\pm 1, \pm 2, \pm 3, \pm 6$

List factors of the leading coefficient (q) 18: $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

2.) Make an organized list of all possible rational roots $\dfrac{p}{q}$

$$\boxed{\frac{\pm 1}{1}, \frac{\pm 1}{2}, \frac{\pm 1}{3}, \frac{\pm 1}{6}, \frac{\pm 1}{9}, \frac{\pm 1}{18}} \qquad \boxed{\frac{\pm 2}{1}, \frac{\pm 2}{2}, \frac{\pm 2}{3}, \frac{\pm 2}{6}, \frac{\pm 2}{9}, \frac{\pm 2}{18}}$$

$$\boxed{\frac{\pm 3}{1}, \frac{\pm 3}{2}, \frac{\pm 3}{3}, \frac{\pm 3}{6}, \frac{\pm 3}{9}, \frac{\pm 3}{18}} \qquad \boxed{\frac{\pm 6}{1}, \frac{\pm 6}{2}, \frac{\pm 6}{3}, \frac{\pm 6}{6}, \frac{\pm 6}{9}, \frac{\pm 6}{18}}$$

3.) Eliminate any double numbers, i.e. $\dfrac{6}{18}$ is the same as $\dfrac{3}{9}, \dfrac{2}{6},$ and $\dfrac{1}{3}$. Leave the $\dfrac{1}{3}$ and cross off the others. Continue with this process to eliminate any other double numbers.

$$\boxed{\frac{\pm 1}{1}, \frac{\pm 1}{2}, \frac{\pm 1}{3}, \frac{\pm 1}{6}, \frac{\pm 1}{9}, \frac{\pm 1}{18}} \qquad \boxed{\frac{\pm 2}{1}, \cancel{\frac{\pm 2}{2}}, \frac{\pm 2}{3}, \cancel{\frac{\pm 2}{6}}, \frac{\pm 2}{9}, \cancel{\frac{\pm 2}{18}}}$$

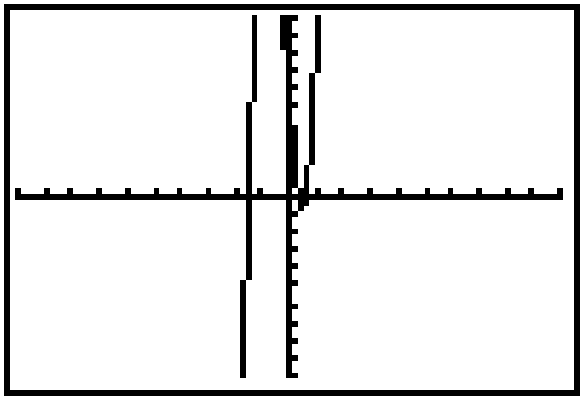
$$\boxed{\frac{\pm 3}{1}, \frac{\pm 3}{2}, \cancel{\frac{\pm 3}{3}}, \cancel{\frac{\pm 3}{6}}, \cancel{\frac{\pm 3}{9}}, \cancel{\frac{\pm 3}{18}}} \qquad \boxed{\frac{\pm 6}{1}, \cancel{\frac{\pm 6}{2}}, \cancel{\frac{\pm 6}{3}}, \cancel{\frac{\pm 6}{6}}, \cancel{\frac{\pm 6}{9}}, \cancel{\frac{\pm 6}{18}}}$$

4.) List all possible remaining roots

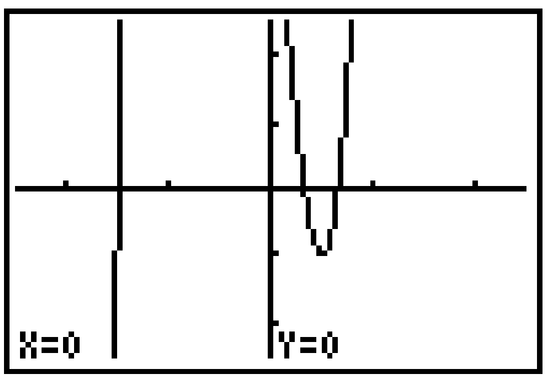
$$\frac{\pm 1}{1}, \frac{\pm 1}{2}, \frac{\pm 1}{3}, \frac{\pm 1}{6}, \frac{\pm 1}{9}, \frac{\pm 1}{18}, \frac{\pm 2}{1}, \frac{\pm 2}{3}, \frac{\pm 2}{9}, \frac{\pm 3}{1}, \frac{\pm 3}{2}, \frac{\pm 6}{1}$$

5.) Graph $18x^3 + 9x^2 - 23x + 6$ on your graphing calculator:

Y = 18 XTθN ∧ 3 + 9 XTθN x^2 − 23 XTθN + 6 (Press ZOOM 6)



6.) By using the graph on your calculator, pinpoint where the graph crosses the *x*-axis. Use ZOOM 2 ENTER to zoom in the screen.



7.) Note where the graph crosses the *x*-axis and refer back to the choices of solutions: $\frac{\pm1}{1}, \frac{\pm1}{2}, \frac{\pm1}{3}, \frac{\pm1}{6}, \frac{\pm1}{9}, \frac{\pm1}{18}, \frac{\pm2}{1}, \frac{\pm2}{3}, \frac{\pm2}{9}, \frac{\pm3}{1}, \frac{\pm3}{2}, \frac{\pm6}{1}$. Eliminate any solutions that are not possible using the graph.

8.) Select a possible solution and use synthetic division or the factor theorem to test your selection. Remember a remainder of 0 in synthetic division means the number is a root. Let's try $\frac{-3}{2}$, as it looks like the graph crosses the *x*-axis at $\frac{-3}{2}$.

$\frac{-3}{2}$ |	18	9	–23	6
		–27	27	–6
	18	–18	4	0

Since the remainder is 0, this means that $\frac{-3}{2}$ is a zero.

From the graph on the calculator, there are two *x*-intercepts between 0 and +1. Our choices from the possibilities are $\frac{\pm 1}{2}, \frac{\pm 1}{3}, \frac{\pm 1}{6}, \frac{\pm 1}{9}, \frac{\pm 1}{18}, \frac{\pm 2}{3}, \frac{\pm 2}{9}$. You can either use the TRACE feature on your calculator to pinpoint the solutions closer, or you can use your own judgment and try synthetic division or the Factor Theorem. From the looks of the graph, the *x*-intercepts are probably $\frac{1}{3}$ and $\frac{2}{3}$. This time test the results using the Factor Theorem:

$$P(x) = 18x^3 + 9x^2 - 23x + 6$$

$$P(x) = 18\left(\frac{1}{3}\right)^3 + 9\left(\frac{1}{3}\right)^2 - 23\left(\frac{1}{3}\right) + 6$$

$$P(x) = 18\left(\frac{1}{27}\right) + 9\left(\frac{1}{9}\right) - 23\left(\frac{1}{3}\right) + 6$$

$$P(x) = \left(\frac{18}{27}\right) + \left(\frac{9}{9}\right) - \left(\frac{23}{3}\right) + 6$$

$$P(x) = \left(\frac{2}{3}\right) + \left(\frac{3}{3}\right) - \left(\frac{23}{3}\right) + \frac{18}{3}$$

$$P(x) = \frac{2 + 3 - 23 + 18}{3} = \frac{0}{3} = 0$$

$$P(x) = 18x^3 + 9x^2 - 23x + 6$$

$$P(x) = 18\left(\frac{2}{3}\right)^3 + 9\left(\frac{2}{3}\right)^2 - 23\left(\frac{2}{3}\right) + 6$$

$$P(x) = 18\left(\frac{8}{27}\right) + 9\left(\frac{4}{9}\right) - 23\left(\frac{2}{3}\right) + 6$$

$$P(x) = \left(\frac{16}{3}\right) + \left(\frac{12}{3}\right) - \left(\frac{46}{3}\right) + 6$$

$$P(x) = \left(\frac{16}{3}\right) + \left(\frac{12}{3}\right) - \left(\frac{46}{3}\right) + \frac{18}{3}$$

$$P(x) = \left(\frac{16 + 12 - 46 + 18}{3}\right) = \frac{0}{3} = 0$$

Both results are zero, which tells us that $\frac{1}{3}$ and $\frac{2}{3}$ are both factors along with the $\frac{-3}{2}$ we found at the beginning of the example.