CONVERGENT AND DIVERGENT INFINITE SERIES: MATHEMATICAL INDUCTION AND THE BINOMIAL THEOREM

In the last unit we examined techniques for establishing the limits of sequences and used a few examples of infinite series to illustrate the increased complexity of problems that deal with infinite processes. In this unit we will examine infinite series in more detail and develop techniques for establishing their sums.

Infinite Series: The Ratio Test

As stated in the last unit, infinite series may actually sum to a final value. Their sums can be infinite, or they can be inconclusive. When a series sums to a final value, then the series "converges" or is "convergent". When a series sums to infinity or is inconclusive, then the series "diverges" or is "divergent".

Ratio Test for Infinite Series: Let a_n and a_{n+1} be two consecutive terms of a positive series. Suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ where $r \in \mathbb{R}$. Then the series converges if $r < 1$; diverges if $r > 1$ and the series may or may not converge if $r = 1$.

It is important to note that the value of "r" **is not the sum** of the series. This value only tells you that the series converges but not to what value.

Example #1: Use the ratio test to determine if the following series converges or diverges.

$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \dots\dots\dots + \frac{n}{3^n} + \dots\dots$$

Step #1: Find the general expression for the a_n and a_{n+1} terms of the series. (Use techniques from previous units to assist in this process when the general pattern is not provided.) In this example $a_n = \frac{n}{3^n}$: Therefore the a_{n+1} term is,

$$a_{n+1} = \frac{n+1}{3^{n+1}}.$$

Step #2: Find the ratio of the two terms and simplify.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \frac{n+1}{\cancel{3}^n \cdot 3} \cdot \frac{\cancel{3}^n}{n} = \frac{n+1}{3n}$$

Step #3: Find the limit of the ratio.

$$\lim_{n\to\infty} \frac{n+1}{n \cdot 3} = \frac{1}{3} = r$$

(Clearly the denominator and numerator increase at the same rate as $n \to \infty$ because the "+1" term contributes nothing to the result at ∞)

Step #4: Conclusion: Since $r = \frac{1}{3} < 1$ the series converges.

Infinite Series: The Polynomial Quotient Test

Polynomial Quotient Test for Convergence (PQT)

Suppose A_o and B_o are polynomial expressions of the forms:

$$P_o(A_o) = a_o x^k + a_1 x^{k-1} + a_2 x^{k-2} + ... + a_{k-2} x^2 + a_{k-1} x + a_k x^0$$

$$Q_o(B_o) = b_o x^n + b_1 x^{n-1} + b_2 x^{n-2} + ... + b_{n-2} x^2 + b_{n-1} x + b_n x^0$$

where $n,\ k \in \mathbb{R} : a,\ b \in \mathbb{R}$.

Then, for $\frac{P_o}{x^n}$ and $\frac{Q_o}{x^n}$ the series converges for:

$$\lim_{k\to\infty} \frac{\frac{P_o}{x^n}}{\frac{Q_o}{x^n}} = \frac{A_k}{B_n} = \frac{a_0 x^{k-n}}{b_0 x^0} + \frac{\frac{a_1}{x^1} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \ . . . + \frac{a_k}{x^{k-n}}}{\frac{b_1}{x^1} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \frac{b_4}{x^4} + \ . . . + \frac{b_n}{x^n}} = \frac{a_0 x^{k-n}}{b_0 x^0}$$

provided $k - n \le 0$; otherwise the series diverges.

Although this theorem appears quite complicated, its application to series convergence is quite useful and not too difficult to determine as the following examples illustrate; but first, a special notation is described to indicate series.

To denote a series of any form, we use the Geek letter **"Sigma"** which appears as:

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + ...a_k$$

If the series is infinite we have $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...a_k + ...$

The bottom notation $"n=1"$, indicates where the series begins and the top notation, $(k \text{ or } \infty)$, where the series ends.

For example:

$$\sum_{n=5}^{8} \frac{1}{n^2} = \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} \quad \text{which when evaluated} = 0.1038 .$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot 2 = 2 - 2 + 2 - 2 + 2 - 2 + ... \text{ which we learned in a previous unit diverges.}$$

Now that we have notation for series, we can use the Polynomial Quotient Test (PQT) in the following examples.

Determine if the following series converge or diverge.

Example #1: $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 2n}{4n^2 + 5}$

Step #1: By the PQT $P_o(A) = 3n^2 + 2n$ and $Q_o(B) = 4n^2 + 5$ and $k = 2 = m$. (For the highest exponent on $"n"$ in each polynomial – use $"m"$ for the exponent on $Q_o(B)$ so that there is no confusion).

a) Find $\dfrac{P_o(A)}{n^2} = \dfrac{3n^2}{n^2} + \dfrac{2n}{n^2} \quad = \quad 3 + \dfrac{2}{n}$.

b) Find $\dfrac{Q_o(A)}{n^2} = \dfrac{4n^2}{n^2} + \dfrac{5}{n^2} \quad = \quad 4 + \dfrac{5}{n^2}$.

Step #2: Find $\displaystyle\lim_{n\to\infty} \frac{\frac{P_o(A)}{n^2}}{\frac{Q_o(B)}{n^2}} \quad = \quad \frac{3 + \frac{2}{n}}{4 + \frac{5}{n^2}}$.

Clearly as $n \to \infty$ “$+\dfrac{2}{n}$” and “$+\dfrac{5}{n^2}$” each come closer to zero. Therefore, at ∞ the expression becomes $\dfrac{3+0}{4+0} = \dfrac{3}{4}$; hence, the series converges.

Example #2: $\sum_{n=1}^{\infty}\frac{2n^2+6n-7}{5n^3+4n+22}$

Step #1: $P_0(A)=2n^2+6n-7 \qquad Q_0(B)=5n^3+4n+22$

$\Rightarrow \;\; k=2$ and $m=3$

Step #2: Since $k<m,$ we will divide both numerator and denominator by n^3 which is the highest power term in the denominator which is stated in the theorem as:

$\dfrac{\frac{P(A)_o}{x^n}}{\frac{Q_o(B)}{x^n}}$ even though the highest power on $P_o(A)$ is k .

The result of this division is:

$$\sum_{n=1}^{\infty}\frac{\frac{2}{n}+\frac{6}{n^2}-\frac{7}{n^3}}{5+\frac{4}{n^2}+\frac{22}{n^3}}$$

Clearly as $n\to\infty$ every term in the expression except the "5" in the denominator approaches zero which gives

$\sum_{n=1}^{\infty}\frac{0+0+0}{5+0+0}=\frac{0}{5}=0$. Therefore the series converges.

Example #3: Find the value of $\sum_{n=1}^{\infty}\frac{4n^2-6}{\sqrt{n^4+5n^2-1}}$.

Step #1: $P_o(A)=4n^2-6 \qquad \Rightarrow k=2$

$Q_o(B)=n^4+5n^2-1 \qquad \Rightarrow m=4$

However ,the presence of the "$\sqrt{\;}$" symbol calls for a bit more analysis. Recall that given the inequality $a>b \;\; \Rightarrow \;\; \frac{1}{a}<\frac{1}{b}$.

Consider then $n^4+5n^2-1 \;>\; n^4 \quad \Rightarrow \quad \frac{1}{n^4+5n^2-1} \;<\; \frac{1}{n^4}$

Which therefore implies that $\dfrac{1}{\sqrt{n^4+5n^2-1}} < \dfrac{1}{\sqrt{n^4}} = \dfrac{1}{n^2}$.

Hence $m=2$ and not $m=4$ as stated above, or, stated differently:

Step #2: Find $$\frac{\frac{P_o(A)}{n^m}}{\frac{Q_o(B)}{n^m}} = \frac{\frac{4n^2-6}{n^2}}{\frac{\sqrt{n^4+5n^2-1}}{\sqrt{n^4}}} = \frac{4-\frac{6}{n^2}}{\sqrt{\frac{n^4+5n^2-1}{n^4}}} = \frac{4-\frac{6}{n^2}}{\sqrt{1+\frac{5}{n^2}-\frac{1}{n^4}}}$$

Now find: $$\lim_{n\to\infty} \frac{4-\frac{6}{n^2}}{\sqrt{1+\frac{5}{n^2}-\frac{1}{n^4}}} = \frac{4-0}{\sqrt{1+0-0}} = \frac{4}{\sqrt{1}} = 4$$

Therefore the series converges.

Example #4: Find the convergence or divergence of: $\displaystyle\sum_{n=1}^{\infty}\frac{9-5n^4}{6n^3+5n^2+n}$.

Step #1: $P_o(A) = 9-5n^4 \quad \Rightarrow \quad k=4$

$Q_o(B) = 6n^3+5n^2+n \quad \Rightarrow \quad m=3$

Since $k \not\leq m$, the series diverges.

or

$$\frac{\frac{P_o(A)}{n^3}}{\frac{Q_o(B)}{n^3}} = \frac{\frac{9}{n^3}-5n}{6+\frac{5}{n}+\frac{1}{n^2}},$$ and

$$\lim_{n\to\infty}\frac{\frac{9}{n^3}-5n}{6+\frac{5}{n}+\frac{1}{n^2}} = \frac{0-5n}{6+0+0} = \frac{-5n}{6} = -\infty$$

Therefore the series diverges.

Infinite Series: Combination Tests

The **Ratio Test** can be used in combination with the **Polynomial Quotient Test (PQT)** in the following manner.

Example #1: Find the convergence or divergence of; $\sum_{n=1}^{\infty} \frac{2n+7}{2^n(n+1)}$.

Step #1: For the Ratio Test, the n^{th} term is $a_n = \frac{2n+7}{2^n(n+1)}$ and the $n+1$ term is

$$\frac{2(n+1)+7}{2^{n+1}(n+1+1)} = \frac{2n+2+7}{2^{n+1}(n+2)} = \frac{2n+9}{2^{n+1}(n+2)} = a_{n+1}$$

$$\Rightarrow \text{Comparing : } \frac{a_{n+1}}{a_n} = \frac{\dfrac{2n+9}{2^{n+1}(n+2)}}{\dfrac{2n+7}{2^n(n+1)}} = \frac{2^n(n+1)(2n+9)}{2^{n+1}(n+2)(2n+7)}$$

$$= \frac{2^n(n^2+11n+9)}{2^n \cdot 2(2n^2+11n+14)} = \frac{n^2+11n+9}{4n^2+22n+28}$$

Step #2: $P_o(A) = n^2+11n+9 \;\; and \;\; Q_o(B) = 4n^2+22n+28$

$$\Rightarrow k = m = 2$$

Step #3:

$$\frac{\dfrac{P(A)_o}{n^2}}{\dfrac{Q_o(B)}{n^2}} = \frac{1+\dfrac{11}{n}+\dfrac{9}{n^2}}{4+\dfrac{22}{n}+\dfrac{28}{n^2}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1+\dfrac{11}{n}+\dfrac{9}{n^2}}{4+\dfrac{22}{n}+\dfrac{28}{n^2}} = \frac{1+0+0}{4+0+0} = \frac{1}{4}$$

Therefore the series converges.

Example #2: Determine if the following series converges.

$$a_n = \frac{\left(\frac{2}{5}\right)^n (5n^2)}{6n+5}$$

By the Polynomial Quotient Test $k = 2,\ m = 1,$ therefore the series would diverge. However the presence of the $\left(\frac{2}{5}\right)^n$ term may have an effect on this outcome, so again the Ratio Test comes into play.

Step #1: $a_n = \frac{\left(\frac{2}{5}\right)^n (5n^2)}{6n+5}$ $\qquad a_{n+1} = \frac{\left(\frac{2}{5}\right)^{n+1} (5(n+1)^2)}{6(n+1)+5}$

Step #2: Simplify a_{n+1}.

$$a_{n+1} = \frac{\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^n (5(n^2+2n+1))}{6n+11} = \frac{\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^n (5n^2+10n+5)}{6n+11}$$

Step #3: Find the Ratio of $\frac{a_{n+1}}{a_n}$.

$$\frac{\frac{2}{5}\left(\frac{2}{5}\right)^n (5n^2+10n+5)}{6n+11} \div \frac{\left(\frac{2}{5}\right)^n (5n^2)}{6n+5} = \frac{\frac{2}{5}\left(\frac{2}{5}\right)^n (5n^2+10n+5)}{6n+11} \times \frac{6n+5}{\left(\frac{2}{5}\right)^n (5n^2)}$$

$$\frac{\frac{2}{5}(5n^2+10n+5)(6n+5)}{5n^2(6n+11)} = \frac{(2n^2+4n+2)(6n+5)}{30n^3+55n^2}$$

Multiplying the numerator using algebra techniques gives the expression:

$$(2n^2+4n+2)(6n+5) = 12n^3+34n^2+32n+10$$

$$\frac{a_{n+1}}{a_n} = \frac{12n^3+34n^2+32n+10}{30n^3+55n^2}$$

Step #4: Use the PQT $k = m = 2$.

$$\lim_{n\to\infty} \frac{12 + \frac{34}{n} + \frac{32}{n^2} + \frac{10}{n^3}}{30 + \frac{55}{n}} = \frac{12}{30} = \frac{2}{5}$$

Therefore the series converges.

Up to this point, all that has been introduced is to test whether an infinite series converges or diverges; but, no actual sum has been calculated. As has been noted over the last three units, many series do sum to a finite value that can be calculated. However techniques, which find actual sums, can be quite complicated and are better left for another course. For the remainder of this unit, we will examine two topics that play an important role in higher mathematics and extend the techniques and concepts learned thus far. These topics are the **Principle of Math Induction** and the **Binomial Theorem**.

Principle of Math Induction

For the past three units, we have often found single expressions that calculate the "n^{th}" term in a sequence or expressions that sum a series. Yet when dealing with infinite processes, it is sometimes difficult to be sure if the expression chosen actually represents all terms even as n becomes quite large or infinite. Although infinity can be assessed and analyzed to a certain degree, no one has actually dealt with its meaning or value in quite the same way that we deal with a simple sum such as "2 + 3" and etc. In fact, calculations on even the largest values by the most sophisticated computers still leave an infinite number of calculations left undone. At best, values of infinite processes are merely "predictions" on those values based upon the soundest logical processes available. The most commonly used technique to logically determine that a chosen or observed pattern or process will continue to infinity is the **Principle of Math Induction (PMI)**. Simply stated, the **PMI** operates on the following three premises:

1.) An actual pattern (or formula) can be determined for the first "n" terms in a sequence or series.

2.) The determined pattern can be tested for the next "$n+1$" term.

3.) If the pattern continues for this next term. then we conclude that the pattern will continue for the remaining infinite number of terms.

Actually determining the formula to be tested by PMI can be quite difficult and will not be dealt with in this unit. In example #2 in the unit link to "Infinite Series: Combination Tests", the first five terms of the series are:

$$\frac{2}{11}+\frac{16}{85}+\frac{72}{575}+\frac{256}{3625}+\frac{32}{875}+.....+\frac{\left(\frac{2}{5}\right)^n(5n^2)}{6n+5}+....$$

To determine this formula, the pattern is not readily apparent in the first five terms and may not even be apparent for the first 1000 terms or more. Therefore, the examples and exercises for this material will simply test a pattern that is already determined.

The procedure for testing patterns by PMI is as follows:

Step #1: Verify the formula works for $n=1$.

Step #2: Assume the formula works for $n=k$.

Step #3: Demonstrate the formula works for $n=k+1$.

The following examples demonstrate this process.

Example #1: Use PMI to verify that:

$$1+2+3+4+\ ...\ +n=\frac{n(n+1)}{2}$$

(The formula to be verified is on the right side of the equals sign).

Step #1: Demonstrate the formula works for $n=1$.

For $n=1$, the first term in the series is "1". Therefore we need to show that

$$\frac{n(n+1)}{2}=1 \quad \text{when } n=1.$$

$$\Rightarrow \quad \frac{1(1+1)}{2}=1$$

$$\frac{1(2)}{2}=\frac{2}{2}=1 \quad \textit{verified}$$

Step #2: Assume the formula works for $n=k$, or that

$$1+2+3+4+\ ...\ +k=\frac{k(k+1)}{2}$$

Step #3: Demonstrate the formula works for $n=k+1$.

For $n=k+1$ we have:

$$1+2+3+4+\ ...\ +k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$

The next task is simplify the right side of the equation and show that the left side returns the same result.

Step #4: Simplify the right side.

$$\frac{(k+1)((k+1)+1)}{2}=\frac{(k+1)(k+2)}{2}$$

Step #5: Show the left side of the equation equals the right side.

$$1+2+3+4+\ \ldots\ +k+(k+1)=\frac{(k+1)(k+2)}{2}$$

From step #2 we know that

$$1+2+3+4+\ \ldots\ +k=\frac{k(k+1)}{2}$$

This value is now substituted into step #3 in the following manner.

$$\underbrace{1+2+3+4+\ \ldots\ +k}_{\frac{k(k+1)}{2}}$$

$$\Rightarrow \quad \frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$$

We now simplify and factor the left side.

$$\frac{k(k+1)}{2}+(k+1)=\frac{k^2+k}{2}+\frac{2k+2}{2}$$

$$=\frac{k^2+3k+2}{2}$$

$$=\frac{(k+1)(k+2)}{2}$$

Since the left side does simplify to the right side, we conclude that the formula for calculating the sum of the first "n" natural numbers is $\frac{n(n+1)}{2}$.

Developing an intuitive sense for PMI often takes a great deal of time and practice to convince oneself that the procedure actually works and is not simply a “trick” of algebra as you may be thinking at this time.

Example #2: Use PMI to show that:

$$1+4+7+10+\ \ldots\ +(3n-2)=\frac{n(3n-1)}{2}$$

Step #1: Show true for $n=1$.

$$a_1=1 \quad \Rightarrow \quad \frac{1(3\cdot 1-1)}{2}=\frac{1\cdot(3-1)}{2}=\frac{1\cdot 2}{2}=1$$

Step #2: Accept true for $n=k$.

$$\Rightarrow \quad 1+4+7+10+\ \ldots\ +(3k-2)=\frac{k(3k-1)}{2}$$

Step #3: Demonstrate for $n=k+1$.

$$1+4+7+10+\ \ldots\ +(3k-2)+(3(k+1))-2=\frac{(k+1)(3(k+1)-1)}{2}$$

Step #4: Simplify the right side.

$$\frac{(k+1)(3(k+1)-1)}{2}=\frac{(k+1)(3k+2)}{2}$$

Step #5: Demonstrate that the left side equals the right side by using substitution of step#2 into step #3.

$$\underbrace{1+4+7+10+\ \ldots\ +(3k-2)}_{\frac{k(3k-1)}{2}}+(3k+1)=\frac{(k+1)(3k+2)}{2}$$

$$\Rightarrow \quad \frac{k(3k-1)}{2}+(3k+1)=\frac{(k+1)(3k+2)}{2}$$

$$\frac{3k^2-k}{2}+\frac{6k+2}{2}=\frac{3k^2+5k+2}{2}$$

$$\frac{(k+1)(3k+2)}{2} \Rightarrow \ \textit{Verified}$$

Example #3: Use PMI to show that:

$$1^2 + 3^2 + 5^2 + 7^2 + \ ... \ + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Step #1: Show true for $n = 1$.

$$1 = \frac{1(2 \cdot 1 - 1)(2 \cdot 1 + 1)}{3}$$

$$1 = \frac{1(2-1)(2+1)}{3}$$

$$1 = \frac{1(1)(3)}{3} = \frac{3}{3} = 1$$

Step #2: Assume true for $n = k$.

$$1^2 + 3^2 + 5^2 + 7^2 + \ ... \ + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

Step #3: Show true for $n = k+1$.

$$1^2 + 3^2 + 5^2 + 7^2 + \ ... \ + (2k-1)^2 + (2(k+1)-1)^2 = \frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3}$$

Step #4: Simplify the right side of the equation.

$$\frac{(k+1)(2(k+1)-1)(2(k+1)+1)}{3} = \frac{(k+1)(2k+1)(2k+3)}{3}$$

Step #5: Substitute the right side of Step #2 into Step #3, and then use algebra to show that the left side equals the simplified expression in Step #4.

$$\underbrace{1^2+3^2+5^2+7^2+\ ...\ +(2k-1)^2}_{\frac{k(2k-1)(2k+1)}{3}}+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}$$

$$\frac{(k)(2k-1)(2k+1)}{3}+(2k+1)^2=\frac{(k+1)(2k+1)(2k+3)}{3}$$

$$\frac{k(4k^2-1)}{3}+\frac{3(2k+1)(2k+1)}{3}=$$

$$\frac{4k^3-k}{3}+\frac{12k^2+12k+3}{3}=$$

$$\frac{4k^3+12k^2+11k+3}{3}=$$

Step #6: Multiply the right side fully (recall techniques from algebra).

$$\frac{(k+1)(2k+1)(2k+3)}{3}=\frac{(k+1)(4k^2+8k+3)}{3}$$

Expand the numerator.

$$\begin{array}{r} 4k^2+8k+3 \\ \times \qquad k+1 \\ \hline \end{array} \quad = \quad 4k^3+12k^2+11k+3$$

and

$$\frac{4k^3+12k^2+11k+3}{3}=\frac{4k^3+12k^2+11k+3}{3} \Rightarrow \textit{Verified}$$

The Binomial Theorem

Recall from algebra that a binomial is an expression of the form:

$$(a+b)^n \text{ where } n \in \mathbb{N} \;\;\&\;\; a,\ b \in \mathbb{R}$$

In algebra you learned to expand binomials when $n = 2,\ 3,\ 4$ or maybe $n = 5$, such as:

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= (x+y)(x^2+2xy+y^2)\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

However, expanding a binomial for large values of n would be quite time consuming. Fortunately, there is a theorem and a device that gives a pattern for expanding binomials for any value of n. The theorem is called the **Binomial Theorem** and the device applied to this theorem is called **Pascal's Triangle**.

Binomial Theorem: Let $(a+b)^n$ be a binomial where $n \in \mathbb{N}$ & $a,\ b \in \mathbb{R}$; then, the "k^{th}" term of the binomial expansion is given by;

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \ .\ .\ .\ + b^n$$

Notice in this expression that the exponent on a decreases while the exponent on b increases by one for each new term.

The **Binomial Theorem** has many applications including the study of Probability where it is used to find such things as the probability of choosing three red marbles in succession from a bag filled with 20 red and 40 blue marbles. Connected to the **Binomial Theorem** is the famous "Pascal's Triangle", which is given below and can be used to find the coefficients of a binomial expansion.

$$\begin{array}{lccccccccccccc}
(x+y)^0 = & & & & & & & 1 & & & & & & \\
(x+y)^1 = & & & & & & 1 & & 1 & & & & & \\
(x+y)^2 = & & & & & 1 & & 2 & & 1 & & & & \\
(x+y)^3 = & & & & 1 & & 3 & & 3 & & 1 & & & \\
(x+y)^4 = & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
(x+y)^5 = & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
(x+y)^6 = & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Notice how the numbers in the row above determines the numbers in each row of the triangle. For example the number "2" in row #3 is the sum of $1+1$ in row #2. The number "4" in row #5 is the sum of $1+3$ or $3+1$ in row #4 and so on. These numbers give the coefficients of an expanded binomial as the following examples show.

Example #1: Use Pascal's Triangle to expand the following binomial.

$$(x+y)^4$$

Step #1: From Pascal's Triangle, $n=4$, the exponent, corresponds to the numbers found in row five. These numbers are; $1 \ 4 \ 6 \ 4 \ 1$.

Step #2: Apply these numbers as the coefficients of the terms of the expanded polynomial. Also note that according to the binomial theorem that the exponent on x will decrease and the exponent on y will increase.

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example #2: Use Pascal's Triangle to expand the following binomial.

$$(5x+6y)^7$$

Step #1: Expand Pascal's Triangle to the 8^{th} row.

Row #6:	1 5 10 10 5 1
Row #7:	1 6 15 20 15 6 1
Row #8:	1 7 21 35 35 21 7 1

Step #2: Use row #8 as the coefficients for the expansion of the binomial and apply descending exponents on the entire first term, "$5x$", and increasing (or ascending) exponents on the entire 2^{nd} term, "$6y$" as follows.

$$(5x+6y)^7 = (5x)^7 + 7\cdot(5x)^6(6y) + 21\cdot(5x)^5(6y)^2 + 35\cdot(5x)^4(6y)^3 + 35\cdot(5x)^3(6y)^4 +$$
$$+ 21\cdot(5x)^2(6y)^5 + 7\cdot(5x)(6y)^6 + (6y)^7$$

We will not apply the exponents to each term as it is obvious that very large numbers will result quite fast. For example the second to last term in the answer if expanded will be: $7\cdot(5x)(6y)^6 = 1632960xy^6$.