SEQUENCES AND SERIES

As we come to the final few units in this course, we will examine mathematical topics that later become the basis for the study of Calculus. Calculus is sometimes called the "Mathematics of Change". In the study of this discipline, it is important to note that the idea of change is towards a specific goal; such as, a final value or a specific function. In our units on exponential and logarithms, we used a couple of these techniques. First we used the procedure of "successive approximations" and the calculator to obtain an approximate value for an equation such as "17 = 5^*x*" to estimate a value for "*x*" as an exponent to within a desired degree of decimal accuracy. In the last unit we briefly introduced the idea of a "limit" to evaluate the numerical value of the base "*e*" from the expression: "*e* = (1+(1/*n*))^*n* as "*n*" approaches infinity.

In the remaining units, we will examine these and other concepts in greater detail. We first begin by looking at **sequences and series** and how these tools enable us to give precise meaning to certain patterns.

Basic Descriptions

The numbers 1, 2, 3, 4, ... (the **Natural Numbers**) provide the most basic example of a "sequence". In general a sequence is simply a list of numerical values to which we can assign a rule that will determine the next number in the list. In the above example, the rule for predicting the next number in the list is:

$$n = k + 1$$

where both n and $k \in \mathbb{N}$

Usually a sequence begins with a first term, k . This, however, is not always the case, as for the sequence:

$$. . ., \ -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3, \ . . . \quad \text{(the **Integers**)}$$

there is no first term, yet the same rule:

$n = k + 1$ for $n, k \in \mathbb{Z}$, still applies.

For those sequences that do not have a definite first term, we often logically or arbitrarily choose the first term and proceed through the remainder of the list in order to ascertain its pattern.

A **"series"** is related to a sequence with one important concept included; the operation of **addition** to sum all the terms in the list. The simplest example of a series is:

$$1 + 2 + 3 + 4 + \ . . .$$

The rule used for a sequence is mainly needed in order to predict the next value in the list. The rules that govern series are needed to determine if the final sum is infinite or an actual number. In the above example, the series clearly sums to ∞. However it can be shown that the series:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ . . . + \frac{1}{n^2} + \ . . . \quad \text{actually sums to "2".}$$

Because sequences and series play such a pivotal role in calculus and higher mathematics, an introduction to their mathematics is the focus of this unit which will concentrate on two of their more basic forms.

Definition of the Arithmetic Sequence

Arithmetic Sequence: an arithmetic sequence is a sequence where each term, after the first term that is denoted a_1, is equal to the sum of the preceding term or terms and a **"common difference"**, denoted "d". The terms of a simple arithmetic sequence can be represented as:

$$a_1,\ a_2 + d,\ a_3 + 2d,\ a_4 + 3d, \ldots$$

Example #1: Find the next three terms in the following arithmetic sequence.

$$-18, -11, -4,\ 3,\ \ldots$$

Step #1: Determine the common difference between terms.

$$-11 - (-18) = 7$$

$$-4 - (-11) = 7$$

$$3 - (-4) = 7$$

Therefore, $d = 7$.

Step #2: Add the common difference to the last term, then to the result, then again and as often as required.

$$3 + 7 = 10$$

$$10 + 7 = 17$$

$$17 + 7 = 24$$

Therefore the next three terms are 10, 17, 24.

If we list terms in a sequence symbolically, then we obtain the following generalization of any sequence:

$$a_1,\ a_1 + d = a_2,\ a_2 + d = a_3,\ a_3 + d = a_4, \ldots$$

$$\ldots, a_{n-2} + d = a_{n-1},\ a_{n-1} + d = a_n,\ a_n + d = a_{n+1}, \ldots$$

From this we can symbolically identify any term in the sequence as:

$$a_k + d = a_{k+1} \qquad \text{or} \qquad a_k = a_{k+1} - d$$

We can further generalize the last expression if we realize that the common difference in a sequence of n terms is added to the terms of the sequence $n-1$ times. For instance, in the sequence 2, 7, 12, 17, there are 4 terms in the sequence therefore $n = 4$ and the common difference is $d = 5$. Starting with the first term and ending with the last, $d = 5$ is added to the terms a total of 3 times.

$$2 + 5 = 7 \qquad 7 + 5 = 12 \qquad 12 + 5 = 17$$

For a sequence with n terms, the common difference d is added to the terms in a sequence $n-1$ times. From this we obtain the following result:

Arithmetic Recursion: the n th term of an arithmetic sequence is given by the formula:

$$a_n = a_1 + (n-1)d$$

and any a_k term in the sequence is given by:

$$a_k = a_{k+1} - d$$

The above "Recursion" formulas are two of the more significant results in mathematics today. Recursion plays a vital role in computer programming as they allow high speed computers (and even your calculator) to identify, and – in the case of series recursion to sum- large lists of numbers in very short period of time; practically at the speed of light.

Example #2: Find the 30th, 49th and 103rd terms in the following sequence.

3, 14, 25, 36, . . .

Step #1: (a) Identify d, (the common difference).

14 – 3 = 11   25 – 14 = 11   36 – 25 = 11

$$\Rightarrow \quad d = 11$$

(b) Identify n to be used in the recursion formula:

For the 30th term, $n = 30$.

For the 49th term, $n = 49$.

For the 103rd term $n = 103$.

Step #2: Substitute value for n and d into the recursion formula along with $a_1 = 3$, the 1st term in the sequence.

30th term:	**49th term**:	**103rd term**:
$a_n = a_1 + (n-1)d$	$a_n = a_1 + (n-1)d$	$a_n = a_1 + (n-1)d$
$a_{30} = 3 + 29 \cdot 11$	$a_{49} = 3 + 48 \cdot 11$	$a_{103} = 3 + 102 \cdot 11$
$a_{30} = 322$	$a_{49} = 531$	$a_{103} = 1125$

Example #3: Find the first term in the arithmetic sequence whose common difference is $d = -3$, and for which $a_{29} = 2$.

Step #1: In this problem $n = 29$, $d = -3$ and a_1 is unknown. If we substitute our values into the recursion formula then simple algebraic methods will provide the value as follows:

$$a_n = a_1 + (n-1)d$$

$$a_{29} = a_1 + (29-1)(-3)$$

$$2 = a_1 + 28(-3)$$

$$2 = a_1 - 84$$

$$86 = a_1$$

*Note: As in this example not all sequences increase. The next two examples demonstrate other values for d as well.

Example #4: Find the 38th term in the sequence whose common difference is $\frac{\pi}{4}$ and whose first term is $\frac{3\pi}{4}$.

Step #1: $n = 38,\ \ d = \frac{\pi}{4},\ \ a_1 = \frac{3\pi}{4}$

Step #2: Substitute:

$$a_n = a_1 + (n-1)d$$

$$a_{38} = \frac{3\pi}{4} + 37\left(\frac{\pi}{4}\right)$$

$$a_{38} = \frac{3\pi}{4} + \frac{37\pi}{4} = \frac{40\pi}{4} = 10\pi$$

The terms between any two terms in a sequence are called **"Arithmetic Means" (A.M.)**

For example:

In this sequence: 2, 8, 16, 32

8 is the A.M. between 2 and 16, and 16 is the A.M. between 8 and 32. We use this concept in the next example.

Example #5: The first term of a sequence is 3.2 and the last term is 5.9. Find the values of 5 A.M. between these terms.

Step #1: $A_1 = 3.2,\ \ A_n = 5.9$

$n = 7$ (5 A.M. between 3.2 and 5.9 plus the starting and ending terms)

$\Rightarrow A_7 = 5.9$

Step #2: Substitute and solve for d.

$$5.9 = 3.2 + (7-1)d$$
$$2.7 = 6d$$
$$0.45 = d$$

Step #3: Starting with $a_1 = 3.2$, add $d = 0.45$ to find the 5 A.M. terms.

$$a_1 + d = 3.2 + 0.45 = 3.65 = a_2$$
$$a_2 + d = 3.65 + 0.45 = 4.1 = a_3$$
$$a_3 + d = 4.1 + 0.45 = 4.55 = a_4$$
$$a_4 + d = 4.55 + 0.45 = 5.0 = a_5$$
$$a_5 + d = 5.0 + 0.45 = 5.45 = a_6$$

$$\text{A.M.} = \{3.65,\ 4.1,\ 4.55,\ 5.0,\ 5.45\}$$

Definition of an Arithmetic Series

The symbol "S_n" is used to denote the sum of an "Arithmetic Series" (and later a "Geometric Series"). Although the sum of an infinite series is often calculable, the procedures are a bit more complex and will be introduced in the next unit. The proof that the sum of the series presented earlier:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ ...+\frac{1}{n^2}+\ ...\ =2$$

is one such series.

Usually the sum of a series is specified to a given number of terms. In the case of fractional series, usually a sum to a specified degree of decimal accuracy is all that is required. For series that sum to values greater than one, the size of the answer determines how many terms should be added. When a series is summed for a specified number of terms, we call this finding the "n th Partial Sum" of the series.

"nth" Partial Sum of an Arithmetic Series

The sum of the first n terms of an Arithmetic Series is given by:

$$S_n=\frac{n}{2}(a_1+a_n)$$

Example #1: Find the sum of the following series.

$$1+2+3+4+\ ...\ +97+98+99+100$$

Step #1: Identify the variables.

$$n=100,\quad a_1=1,\quad a_{100}=100$$

Step #2: Substitute and evaluate.

$$S_n=\frac{100}{2}(1+100)=50*101=5050$$

Example #2: Find the sum of the first 60 terms of the following series.

$$9+14+19+24+\ ...\ +289+294+299+304$$

Step #1: Identify the variables, and then substitute and evaluate.

$$n=60,\quad a_1=9,\quad a_{60}=304$$

$$S_n=\frac{60}{2}(9+304)=30\times 313=9390$$

Notice in this last example that as long as all the components of the summation formula are known, the terms of the series can have any common difference and not just $d=1$ as in the last example.

Example #3: Justin sells cars at a local dealership and earns $150.00 commission on the first car he sells each month. For each additional car Justin sells during the month, he receives $10.00 more on his commission. How many cars will Justin need to sell to earn a total commission of $3600.00 for the month?

Step #1: Identify the variables.

$$a_1=\$150.00,\quad d=10,\quad S_n=\$3600.00,\quad a_n=?$$

In this example $a_n=?$, which also means $n=?$ and we have two unknowns in the summation formula presented earlier. We therefore need a new formula for this situation, which is:

$$S_n=\frac{n}{2}(2a_1+(n-1)d)$$

This formula is found by substituting the previous result, $a_n=a_1+(n-1)d$, for sequences, into a_n in the summation formula. Using this modified result we obtain:

$$3600 = \frac{n}{2}(2*150 + (n-1)10)$$

$$7200 = n(2*150 + (n-1)10)$$

$$7200 = n(300 + 10n - 10)$$

$$7200 = 290n + 10n^2$$

$$10n^2 + 290 - 7200 = 0$$

$$n^2 + 29n - 720 = 0$$

$$(n+45)(n-16) = 0$$

$$n = -45,\ 16$$

Clearly Justin cannot sell -45 cars; therefore, he needs to sell 16 cars to earn a total commission of $3600.00 for the month.

Geometric Sequences

The following is an example of a "Geometric Sequence".

$$2,\ 2^2,\ 2^3,\ 2^4,\ 2^5 = 2,\ 4,\ 8,\ 16,\ 32$$

Another way to look at this sequence is:

$$a_1 = 2,\ \ a_2 = a_1 \cdot 2,\ \ a_3 = a_2 \cdot 2,\ \ a_4 = a_3 \cdot 2,\ \ a_5 = a_4 \cdot 2$$

In each case the next term in the sequence is the product of the previous term and the constant 2. For a Geometric Sequence, the number that multiplies the present term to give the next result is called the **"Common Ratio"**. The effect of the common ratio is perhaps better seen in the next example.

$$4,\ 4\cdot 3,\ 4\cdot 3^2,\ 4\cdot 3^3,\ 4\cdot 3^4 =$$

$$4,\ 12,\ 36,\ 108,\ 324,\ 972 =$$

$$a_1,\ a_1\cdot 3,\ a_2\cdot 3,\ a_3\cdot 3,\ a_4\cdot 3$$

In this example the common ratio is 3 and is denoted $r = 3$.

Geometric Sequence: A Geometric Sequence is a sequence where each term after the first term, a_1, is the product of the preceding term and the common ratio, r, where $r \neq 0$ or 1. The terms of a geometric sequence can be represented by;

$$a_1,\ \ a_2 = a_1 \cdot r,\ \ a_3 = a_2 \cdot r,\ \ a_4 = a_3 \cdot r,\ \ a_5 = a_4 \cdot r\ \ldots$$

"nth" Term of a Geometric Sequence

The nth term of a Geometric Sequence is given by:

$$a_n = a_1 r^{n-1}$$

Example #1: Find the approximate value of the ninth term of the following geometric sequence.

$$5,\ 2,\ \frac{4}{5},\ \frac{8}{25},\ \frac{16}{125},\ \ldots$$

Step #1: Find the common ratio by dividing each term by the previous term.

$$\frac{a_2}{a_1}=\frac{2}{5},\quad \frac{a_3}{a_2}=\frac{4}{5}\div 2=\frac{2}{5},\quad \frac{a_4}{a_3}=\frac{8}{25}\div\frac{4}{5}=\frac{2}{5},\quad \frac{a_5}{a_4}=\frac{16}{125}\div\frac{8}{25}=\frac{2}{5}$$

$$\text{Therefore } r=\frac{2}{5} \text{ and } n=9$$

Step #2: Substitute and evaluate (use calculator).

$$a_n=a_1r^{n-1}$$

$$a_9=5\left(\frac{2}{5}\right)^8 \qquad \text{Type, } 5(2/5)\wedge 8,\ \boxed{\text{ENTER}}$$

$$a_9\approx 0.0032768$$

Example #2: Find the 10th term of the following geometric sequence.

$$18, -54,\ 162, -486,\ 1458,\ \ldots$$

Step #1: Common Ratio:

$$\frac{a_2}{a_1}=\frac{-54}{18}=-3,\quad \frac{a_3}{a_2}=\frac{162}{-54}=-3,\quad \frac{a_4}{a_3}=\frac{-486}{162}=-3$$

$$\therefore\ r=-3,\quad n=10$$

Step #2: Substitute and evaluate.

$$a_{10}=18(-3)^9=-354,294$$

The following example demonstrates how the natural logarithm can be used to find the common ratio of a geometric sequence.

Example #3: The first term of a geometric sequence is 7 and the 6th term is 21875. What is the common ratio?

Step #1: $a_1=7,\quad a_6=21875,\quad n=6$

Step #2: Substitute and simplify.

$$21875=7r^5 \quad\Rightarrow\quad 3125=r^5$$

Step #3: Take "ln" of both sides.

$$\ln 3125 = 5 \ln r$$

$$\frac{\ln 3125}{5} = \ln r \qquad \text{: Using calculator}$$

$$1.6094 = \ln r$$

Step #4: Use inverse relationship for e^x and $\ln r$.

$$1.6094 = \ln r \iff r = e^{1.6094}$$

Type on your calculator: [2nd], [ln], 1.6094, [ )], [ENTER]

$$r \approx 4.9998 \doteq 5$$

Geometric Series

"nth" Partial Sum of a Geometric Series

The sum of the first n terms of a Geometric Series is given by:

$$S_n = \frac{a_1 - a_1 r^n}{1-r}$$

Example #1: Find the sum of the first 10 terms of the following geometric series.

$$16 - 48 + 144 - 432 + \ ...$$

Step #1: Find r.

$$\frac{a_2}{a_1} = \frac{-48}{16} = -3, \quad \frac{a_3}{a_2} = \frac{144}{-48} = -3 \quad \therefore \quad r = -3, \quad n = 10$$

Step #2: Substitute and evaluate (use calculator).

$$S_n = \frac{a_1 - a_1 r^n}{1-r}$$

$$S_{10} = \frac{16 - 16(-3)^{10}}{1-(-3)} = \frac{16 - 16(-3)^{10}}{4} = 4 - 4(-3)^{10}$$

$$S_{10} = -236{,}192$$

Example #2: The sum of a geometric series is 200,000 for the first 5 terms. If the common ratio is $r = 19$, approximate the value of the first term.

Step #1: Substitute, simplify, and evaluate.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$n = 5, \quad r = 19, \quad S_5 = 200,000$$

$$200,000 = \frac{a_1 - a_1 19^5}{1 - 19}$$

$$-18(200,000) = a_1 - a_1 \cdot 2,476,099$$

$$-3,600,000 = a_1(1 - 2,476,099)$$

$$-3,600,000 = -2,476,098 \cdot a_1$$

$$\frac{-3,600,000}{-2,476,098} = a_1$$

$$a_1 \approx 1.4539$$