THE NATURAL EXPONENTIAL AND LOGARITHM WITH APPLICATIONS

Exponential and logarithmic equations and functions can be used in any base, such as $3^x = 10$, $12 = \log_4 y$, with "*a*" = 3, 4 in these expressions. There is, however, one value of the base for these expressions which is especially significant. It is found and used in a wide variety of real-world situations and it appears in so many areas of mathematics as the natural consequence of normal mathematical investigation and research. This base is denoted by the small case letter "*e*" and is called the **"Natural Base"**. When this base is used in exponential or logarithmic expressions, they are called the **"Natural Exponential"** and **"Natural Logarithmic"** expressions. Because the base "*e*" is such a fundamental value, we will use the first part of this unit to find a numerical approximation of its value and manipulate a few equations that involve this number. For the remainder of the unit, and in the assignment, we will explore a range of real-world applications that involve this number.

Derivation and Definition of the Natural Base

The value of the base "*e*" is a "Transcendental number" and is therefore a non-repeating, non-terminating decimal. The numerical approximation of this number is found in the following definition.

Natural Base (e)**:** Let $x \in \mathbb{R}$, then the "Limit" as x approaches infinity $(x \to \infty)$ of the following expression is the Natural Base (e):

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e \qquad \text{where } x > 0$$

The above expression employs the concept of a **"Limit"** which we will explore in greater detail in the final three units of this course. The "Limit" concept is a fundamental concept to the study of Calculus. For this unit we will consider the expression, $\text{"}\lim_{x\to\infty}\text{"}$ as an approximation of the expression by successively allowing x to take on larger and larger values; such as in the following table:

x	$(1+1/x)^x$	Value of the Expression
0.25	$(1+1/0.25)^{0.25}$	1.49534878…
0.5	$(1+1/0.5)^{0.5}$	1.73205807…
0.75	$(1+1/0.75)^{0.75}$	1.88791566…
1.00	$(1+1/1)^{1}$	2.00000000…
2.00	$(1+1/2)^{2}$	2.25
3.00	$(1+1/3)^{3}$	2.370370370…
10.00	$(1+1/10)^{10}$	2.59374246…
100.00	$(1+1/100)^{100}$	2.7048138294…
999.00	$(1+1/999)^{999}$	2.7169225742…
$x \to \infty$	$\left(1+\frac{1}{x\to\infty}\right)^{x\to\infty}$	2.718281828459…

(Note: In reality x can not actually equal ∞, but as x increases without bound to larger and larger values, the expression approaches an actual value called "e". The value of this number is given below to 25 decimal places):

$$e = 2.1782818284590452353602874\ldots$$

The base "e", written in an exponential equation, obeys all the rules of any exponential equation and its graph is similar to the graphs of exponential expressions. In addition, since this curve is "$1-1$", it has a logarithmic inverse which is called the "Natural Logarithm". Because "e" and its logarithm are especially significant values, they are given their own notation and are set aside as special keys on your calculator. The inverse relation for "e" and its logarithm are given below which is derived from the inverse relationship we have used already previous units.

We know:

$$y = a^x \Leftrightarrow x = \log_a y$$

For $a = e$ we now have:

$$y = e^x \Leftrightarrow x = \ln y$$

Where "ln" is the notation for the natural logarithm.

The ln key is located on the front of your calculator and base "e" can be accessed by typing 2nd, ln. To view the decimal approximation of e type 2nd, ln, 1,), ENTER on your home screen.

$$e \wedge (1) = 2.718281828$$

(Recall that your calculator will display 10 decimal places and store 13 in its memory)

To see graphs of the natural exponential and the natural logarithm type the following into Y=

$$Y_1 = e \wedge (x)$$
$$Y_2 = \ln(x)$$

*Make sure all Plots are off: 2nd, STATPLOTS, 5 (PlotsOff), ENTER
*Reset Window: ZOOM, 6: ZStandard

Using the Natural Exponential and Natural Logarithms in Equations

Example #1: Rewrite the expression $y = e^{-2x}$ as a natural logarithm.

Step #1: From $y = e^{x} \Leftrightarrow x \ln y$

We have: $-2x = \ln y$

$$x = \frac{\ln y}{-2}$$

Example #2: Rewrite $x = \ln(3y+2)$ as a natural exponential equation.

Step #1: Notice that no base is written for $\ln(3y+2)$. However, because the symbol "ln" is specifically set aside for the logarithm of the natural base, the base is understood to be "e". Therefore from the above inverse relationship we have:

$$x = \ln(3y+2) \quad \Rightarrow \quad 3y + 2 = e^{x}$$

$$3y = e^{x} - 2$$

$$y = \frac{e^{x} - 2}{3}$$

Example #3: Solve for x: $4y = e^{3x+2}$

$$4y = e^{3x+2} \quad \Leftrightarrow \quad 3x + 2 = \ln 4y$$

$$3x = -2 + \ln 4y$$

$$x = \frac{-2 + \ln 4y}{3}$$

Example #4: Solve for *y*: $x^2 - 4x + 5 = \ln(y^2 - 6)$

$$x^2 - 4x + 5 = \ln(y^2 - 6) \quad \Leftrightarrow \quad y^2 - 6 = e^{x^2 - 4x + 5}$$

$$y^2 = 6 + e^{x^2 - 4x + 5}$$

$$|y| = \sqrt{6 + e^{x^2 - 4x + 5}}$$

Recall that for any logarithmic expression the domain is $x > 0$; therefore, in the above solution

$$y^2 - 6 > 0 \quad \Rightarrow \quad |y| > \sqrt{6}$$

Famous Equations and Graphs of the Natural Exponential and Natural Logarithm

The use and application of the “Natural” functions extend to nearly every branch of science, finance, probability and statistics and even to psychology and sociology. One of the first areas to apply their use was in the area of finance. Nearly every form of financial and economic analysis depends on formulas that in some way are governed by the “Natural” functions. One of the more common uses of these equations is listed below.

I.) The Interest Rate Formula

A form of this equation has already been demonstrated when we used it to approximate the value of $"e"$ at the beginning of this unit. Banks and financial institutions use a modified version of this equation to calculate interest rates. This formula is:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Where: $A =$ Total amount after interest.

$P =$ Initial amount of money to receive interest.

$r =$ The interest rate.

$n =$ The number of times interest is calculated.

$t =$ The amount of time of the investment.

This formula will also appear in the assignment for this unit and is demonstrated in the following example.

Example #1: Randy invests $5,000.00 in a bank that pays 2.5% interest and calculates his interest every three months. After one year, how much money will Randy have in the bank?

Note: The number of times that a bank calculates your interest over a given time period is called **"compounding"**. If Randy were to receive interest on his money only at the end of the year, this would be called **"simple interest compounding"** or **"annual interest"**. Below are a list of terms used according to the number of times a bank compounds the interest in a given year.

n	**Number of Compounding**	**Compounding Term**
$n = 1$	Once a year	Annually
$n = 2$	Twice a year	Semi-annually
$n = 4$	Four times a year	Quarterly
$n = 12$	Every month	Monthly
$n = 365$	Every day	Daily

In the above example, Randy will receive interest on his money every three months or "Quarterly".

Step #1: Assign values to the variables

$A =$	Amount of money Randy will have at the end of the year.
$P = \$5000.00$	The amount of Randy's initial investment.
$r = 2.5\% = 0.025$	The interest rate.
$n = 4$	The number of times interest is calculated for a quarterly compounding.
$t = 1$	The amount of time Randy will invest his money – one year.

Step #2: Substitute values and calculate.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \rightarrow \quad A = 5000\left(1 + \frac{0.025}{4}\right)^{4 \times 1} = \$5{,}126.18$$

Therefore, at the end of one year Randy will have received $126.18 in interest.

*Note: If Randy had invested his money at simple interest, he would have obtained only \$125.00 in interest. $I = prt$

The difference in the two amounts is accounted for by the fact that once Randy's interest is calculated at the end of every three months, this is added to his principal (P). Then at the end of the next three months, the interest is calculated on his initial investment, (\$50000.00), plus the interest he received at the end of the previous three months. The following table demonstrates how the extra four compoundings benefit Randy as opposed to a simple interest investment.

\$5000.00 initial investment	Simple Interest	Quarterly
End of 1st three months	----	$5000\left(1+\frac{0.025}{4}\right)^1 = 5031.25$
End of 2nd three months	----	$5031.25\left(1+\frac{0.025}{4}\right)^1 = 5062.70$
End of 3rd three months	----	$5062.70\left(1+\frac{0.025}{4}\right)^1 = 5094.34$
End of year	$5000(1+0.025)^1 = 5125.00$	$5094.34\left(1+\frac{0.025}{4}\right)^1 = 5126.18$

*Notice that since Randy's money is being calculated at the end of each quarter, the exponent on the expression stays "1" to reflect the fact that each new amount is treated as a 'new' investment by Randy.

Obviously, if Randy could invest his money in a bank that *compounds monthly*, he would receive *a little more money* in interest as:

$$A = 5000\left(1+\frac{0.025}{12}\right)^{12} = 5126.44 \Rightarrow = +\$0.26$$

If Randy could invest his money in a bank that *compounded daily*, or for $n = 365$, then he would have:

$$A = 5000\left(1+\frac{0.025}{365}\right)^{365} = 5126.57 \Rightarrow = +\$0.13$$

If Randy could invest his money in a bank that applies *continuous compounding*, that bank would use the following formula:

$$A = Pe^{rt}$$

and Randy would receive the following amount at the end of the year:

$$A = 5000(e)^{0.025} = 5000(1.025315) = 5126.58 \Rightarrow +0.01$$

II.) The Normal Distribution or "Bell Curve"

The natural base, "e", plays a fundamental role in the study of **'probability and statistics'**. One of the major concerns for college bound high school students is scoring well on the ACT or SAT exams. Once a student takes either of these tests, their results are often compared to the average or **"mean"** score for all other students in the United States, who also took the same test. Obviously, the higher one scores, the better his or her chances are at attending their college of choice. The statistical distribution of these scores is calculated according to the laws of probability, which assume that most students will score comparable to their peers, while a smaller percentage will score either higher or lower. The formula used to analyze these results is:

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{(-x^2/2)}$$

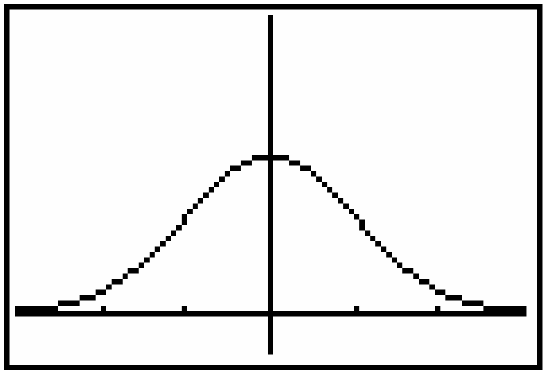
To see the graph of this curve type in the following on your calculator:

Xmin = –3	Ymin = –0.1
Xmax = 3	Ymax = 0.75

$$y_1 = (1/\sqrt{\ }(2\pi)) \cdot e \wedge (-x^2/2)$$

*Note: To obtain e, type [2nd], [ln] and remember to use the [(–)] key for the negative sign.

Your calculator should display the following:



Once the curve is graphed, press TRACE .

You will notice that for $x = 0,\ y = 0.3989$. At the value $x = 0$ the true mean, $(\bar{x})$, of the curve is recorded. This indicates that approximately 40% of all students can be expected to score a true mathematical average. In a typical classroom setting we might expect to find that 50% of all students taking an average math test would score at the mean. It is also significant to note that a true mathematical average involving base e is $\bar{x} = 0$. In our typical classrooms, an “average” grade of “C” is usually recorded as a 75 or 75%. This is due to the fact that grades are given in points, and since no one would accept a negative score for a grade of “C”, certain adjustments are made to satisfy our needs. In the branch of math called Probability, however, a score of “0” indicates the average. (Note also that ACT and SAT scores are also reported as positive numbers. Again this is due to an adjustment to record all scores as positive numbers). In statistics a score to the left of $\bar{x} = 0$ simply indicates a direction from the mean. One common measure used in the study of Statistics is the concept of **“Standard Deviation”**. Although this is not a course in statistics, a brief description of how this measurement is used to analyze data (not just test scores) can be helpful.

A Standard deviation of ± 1 indicates that approximately 68% of all recorded values will fall within this range, to both sides of the mean. A standard deviation of ± 2 will account for about 95% of all scores and a standard deviation of ± 3 will account for about 98% of all data. 100% of all scores will be found under the entire curve. (The actual graph of this curve incorporates these values as well but is beyond the scope of this class. The equation represented on you calculator is a good approximation for our purposes and incorporates the concept that approximately 50% of all data is average).

With your calculator still in TRACE type $x = 1$ and record that $y = 0.242$ now type $x = -1$ and record that $y = 0.242$ again. If we add these two values we see that approximately 48.4% or 50% of all data is within ± 1 standard deviations.

Example #2: On a standard IQ test Rachel scored +1.86 standard deviations above the mean. If 750 students took the test, how many other students scored at Rachel's level?

Step #1: The fact that 750 students took the test is an "initial condition" to the exponential curve now on your calculator. To have your calculator reflect this on the graph, type Y= and retype the equation in Y_1 as follows:

$$\text{Y}_1 = A(1/\sqrt{(2\pi)}) \cdot e \wedge (-x^2/2)$$

We will use "A" to hold our initial condition for this example and for the problems in the assignment.

Step #2: On your home screen type;

750, STO →, ALPHA, A, ENTER

Step #3: Adjust Ymax = 375 and graph. (Since we assume all scores are normally distributed to either side of the mean, we also assume that no more than $\frac{750}{2} = 375$ students will score at the maximum value of the mean).

Step #4: Press TRACE and type $x = 1.86 \Rightarrow y = 53$ students scored the same as Rachel.

III.) The Catenary Curve

In the next description we will briefly examine an important curve in engineering science, particularly in bridge design. The mathematics of the curve, called the **Catenary**, is best explored in Calculus but its basic properties and uses are simply seen.

The equation of the Catenary Function is:

$$f(x) = \frac{e^x + e^{-x}}{2}$$

To see this curve on your calculators enter the following, and then graph using ZOOM, 6.

$$\text{Y}_1 = (e \wedge (x) + e \wedge (-x))/2$$

At first this curve looks parabolic, but despite its similar appearance it is not a parabola. A Catenary curve is the shape assumed by a suspended chain hanging under its own weight. The key to this description is that of a free hanging chain (or rope, cable, etc.) "suspended" at each end. The curve seen on your calculator is the same one formed by the **Golden Gate Bridge** in **San Francisco, California**, and all other suspension bridges. When this curve is inverted (flipped vertically) it is used as a suspension arch, not only in bridges, but also in any structure that needs support from below. The roofs of large cathedrals and some domed stadiums in sports complexes are Catenaries. The curve has the ability to spread the enormous weight of the structure evenly through its span.

IV.) Natural Logarithmic Applications

As the inverse of the Natural Exponential, the Natural Logarithm conforms to all the laws of logarithms and is useful in solving exponential equations in other areas than those that have been previously described, consider the following examples.

Example #3: Suppose in our previous *example #2* that Randy wants to earn $500.00 interest on his investment of $5000.00 and wants to know how long it will take to obtain this amount.

Step #1: Identify the variables in the interest rate formula:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$A = \$5,500.00$: ($5000.00 + $500.00 interest).

$P = \$5,000.00$: (Randy's initial investment).

$r = 0.025$: (Randy's interest rate).

$n = 4$: (quarterly compounding).

$t = ?$: (amount of time, in years, Randy will need to invest).

Step #2: Substitute and simplify.

$$5500 = 5000\left(1+\frac{0.025}{4}\right)^{4t}$$

$$1.1 = \left(1+\frac{0.025}{4}\right)^{4t}$$

Step #3: Raise both sides of the equation to "ln".

$$\ln 1.1 = \ln\left(1 + \frac{0.025}{4}\right)^{4t}$$

Step #4: Simplify using laws of logarithms.

$$\ln 1.1 = 4t\ln\left(1 + \frac{0.025}{4}\right)$$

$$\ln 1.1 \ = \ 4t \ \ln(1.00625)$$

$$t = \frac{\ln 1.1}{4\ln(1.00625)}$$

Step #5: Evaluate the expression on your calculator. Type the following:

[ln], 1.1, [)], / [(], 4, [ln], 1.00625, [)], [ENTER]

$t \approx 3.8$ Years

Since Randy's investment is compounded quarterly, after 3.75 years (3 years, 9 months) Randy's interest will be close to $500.00.

Example #4: Monica purchases 100 shares of stock at $15.00 per share. The company pays dividends on the stock semiannually. If the projected rate of return on Monica's investment is 15%, how long will it take for Monica's stock to grow to a value of $4000.00?

Step #1: Identify the variables

$A = \$4,000.00$ (Amount of money Monica's stock will reach).

$P = \$1,500.00$ ($15.00 per share × 100 shares).

$r = 0.15$ (Interest rate)

$n = 2$ (Semiannual compounding)

$t = ?$ (Time needed for stock value to reach $4,000.00).

Step #2: Substitute and simplify.

$$4000 = 1500\left(1 + \frac{0.15}{2}\right)^{2t}$$

$$\frac{8}{3} = \left(1 + \frac{0.15}{2}\right)^{2t}$$

Step #3: Take ln of both sides of the equation.

$$\ln\frac{8}{3} = \ln\left(1 + \frac{0.15}{2}\right)^{2t}$$

$$\ln\left(\frac{8}{3}\right) = 2t\ln(1.075)$$

$$t = \frac{\ln\left(\frac{8}{3}\right)}{2\ln(1.075)}$$

Step #4: Evaluate using the calculator.

$\boxed{\ln}$, $\frac{8}{3}$, $\boxed{)}$, / $\boxed{(}$, 2, $\boxed{\ln}$, 1.075, $\boxed{)}$, $\boxed{\text{ENTER}}$

$t \approx 6.78$: 6 years, 9 months

Other applications of the natural logarithm are found in science, statistics, and in engineering. The next two examples will explore natural logarithms in these areas.

Example #5: Bacterial growth increases according to the law:

$$N(t) = N_o e^{kt}$$

where N_o is the initial number of bacteria and k is a positive number particular to the strand of bacteria's growth.

If the number of bacteria DOUBLES every 3 hours, find k if the initial number of bacteria is estimated to be 1.75×10^{14} living bacteria in a Petrie dish.

Step #1: Identify the variables.

$N_o = 1.74\times10^{14}$ (Initial amount of bacteria).

$t = 3$ (Time needed for bacteria to double).

$N(t) = 3.5\times10^{14}$ (Amount of bacteria after 3 hours: $2N_o = 3.5\times10^{14}$).

Step #2: Substitute and simplify.

$$3.75\times10^{14} = 1.75\times10^{14}\cdot e^{3k}$$

$$2 = e^{3k}$$

Step #3: Utilize the inverse relationship of exponential and logarithms.

$$y = e^{x} \Leftrightarrow x = \ln y$$

$$2 = e^{3k} \Leftrightarrow 3k = \ln 2$$

$$k = \frac{\ln 2}{3}$$

Step #4: Evaluate using your calculator.

$$k \approx 0.231$$

Example #6: **Newton's Law of Cooling** governs the temperature of a heated object, at a given time. This law states:

$$M(t) = T + (M_o - T)e^{kt}$$

where:

"T" is the constant temperature of the surrounding area (usually room temperature).

"M_o" is the initial temperature of the heated object.

"k" is a negative constant reflecting the physical properties of the object.

An object is heated to $122°C$ (degrees Celsius) and allowed to cool to room temperature which is $28°C$. If the temperature of the object is $80°C$ after 10 minutes, when will its temperature be $50°C$?

Step #1: Identify the variables.

$T = 28^\circ C$ (Room temperature).

$M_o = 122°C$ (Initial temperature of the heated object).

$M(t) = 80°$ (The temperature of the object after time, 10 minutes).

$t = 10$ (Time, in minutes, to cool to $80°C$).

$k = ?$ (Physical constant of the object).

Step #2: Substitute and simplify.

$$M(t) = 28° + (122 - 28)e^{kt}$$

$$M(t) = 28 + 94e^{kt}$$

This is the equation of the cooling function for the object.

Step #3: Determine the value of *k* under the first situation.

$$M(10) = 80 \quad \Rightarrow$$

$$80 = 28 + 94e^{10k}$$

$$52 = 94e^{10k}$$

$$\frac{26}{47} = e^{10k}$$

Step #4: Evaluate k using inverse relationship.

$$10k = \ln\left(\frac{26}{47}\right)$$

$$k = -0.0592$$

Step #5: Evaluate for $M(t) = 50$, find t.

$$50 = 28 + 94e^{-0.0592t}$$

$$22 = 94e^{-0.0592t}$$

$$\frac{11}{47} = e^{-0.0592t}$$

$$-0.0592t = \ln\left(\frac{11}{47}\right)$$

$$t = \frac{\ln\left(\frac{11}{47}\right)}{-0.0592}$$

$t \approx 24.53 \doteq 25$ minutes to cool to $50°C$.