APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponential and logarithmic equations have many uses in the real world. Interest rate calculations on savings, investments, and loans are made using these types of equations. Animal and plant population growth and death rates are calculated and predicted using exponential functions. How fast a liquid cools to room temperature, fluid flow rates through piping systems, random and conditionally random distribution of events are all made using exponential or logarithmic models. In the next two units we will examine some of these applications in a series of problems. We will also define two special functions; the Natural Exponential and the Natural Logarithm and explore how these functions relate to important real-life situations.

Extended Definition of the Exponential and Logarithm

Scatter Plots and Regression Equations (From Unit #13)

Completing the Analysis of the Ping-Pong Ball Problem in Unit #13

Finding the Equations of the Missing Parabolas

Extended Definition of the Exponential and Logarithm

Recall that we gave the expression $y = a^x$ as the standard form of the exponential equation. Although this expression allowed us to explore and learn the properties of exponentials, and later logarithms, this is not the complete description of these expressions. For this unit – and beyond – we will use the following result as we continue to explore these classes of functions.

Standard Expansion of the Exponential Function

Let a, $b \in \mathbb{R}$ and x a real variable. Then the equation

$$f(x) = ab^x$$

is an exponential function with an **initial condition** $"a"$

Standard Expansion of the Logarithmic Function

Let a, $b \in \mathbb{R}$ and x a real variable. Then the equation

$$f(x) = \log_b \left(\frac{y}{a} \right)^c = c[\log_b y - \log_b a]$$

is the logarithmic function with an **initial condition** $"a"$

(Note: the last definition is derived from,

$$y = ab^x \Rightarrow \frac{y}{a} = b^x$$

$$\frac{y}{a} = b^x \Leftrightarrow x = \log_b \left(\frac{y}{a} \right)$$

In the above definition, the phrase "initial condition" is used to express the exponential and logarithmic function as dependent on, and affected by, real number multipliers. To illustrate this point enter the following equations into your calculator to examine how the 'initial conditions" affect the graph of $y = 2^x$.

Type the following into $\boxed{Y=}$ and graph one by one (You may wish to turn functions off and adjust window to see more clearly)

$$Y_1 = 2^x \qquad Y_2 = 3 \cdot 2^x \qquad Y_3 = -5 \cdot 2^x$$

$$Y_4 = x \cdot 2^x \qquad Y_5 = x^2 \cdot 2^x \qquad Y_6 = x^3 \cdot 2^x$$

$$Y_7 = (11x) \cdot 2^x \qquad Y_8 = \cos x \cdot 2^x \qquad Y_9 = (x-3) \cdot 2^x$$

For the last six functions graphed, the initial condition is itself a function. As these functions become more complex, their graphs can model many situations; such as weather fluctuations and heart rates. For those students who will take calculus in the future, the analyses of these types of initial conditions motivate a broad range of application. For this course, we will examine simpler exponential and logarithmic models to real world situations.

Scatter Plots and Regression Equations (From Unit #13)

Using matrices to find the equation of a parabola from a set of three points can be a tedious task, even on the graphing calculator, when those points are decimal values which are extremely large, or extremely small. In addition, finding only one curve may not be all that is required; especially, if the values of our points were acquired in a real-world situation. As you are probably aware from your science courses, data collected in experiments rarely yield whole number or even simple fractional results. Still, the methods of mathematics are expected to provide meaning to this data. In this part of the lesson, we will examine a set of data collected in a real-world experiment and use the graphing calculator to provide mathematical equations to describe this data. When we are done with this lesson, our analysis of the data will be incomplete. Later in the course we will return to this problem after we examine the mathematics of exponential and logarithmic equations. This knowledge will be required to finish our analysis of the information provided in this lesson.

Consider the following situation:

> A ping pong ball is dropped from a height of four feet and allowed to bounce on the floor, uninterrupted, until it comes to rest. Use mathematics to describe the following:
>
> 1. Any one bounce of the ball.
> 2. The series of all bounces of the ball.
> 3. Predict an equation for any one bounce of the ball using information collected from the other bounces.

> In your physics or physical science classes, you may have encountered that an object tossed into the air and allowed to return to the ground will trace a parabolic curve (provided no other force except gravity acts on the path of the object which we will assume for this lesson). In the problem above, our ping pong ball traces a parabolic curve in between each bounce. Because gravity eventually causes the ball to come to rest, each parabola traced by the ball decreases in height at a uniform rate according to a logarithmic curve. (This will be discussed in a future lesson when we return to this data).

> As the ping pong ball travels through the air, between each bounce, it is possible to record its position at many points using a motion detector, the graphing calculator, and a CBL (Calculator Based Laboratory). The information acquired for this lesson came from this equipment. Although this course does not require the use of a CBL, you may have encountered its use in your science or other math courses.

> The series of bounces may all be graphed in sequence and will look similar to the following diagram:
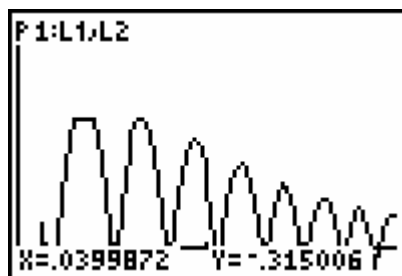
For this diagram the height of the ball's bounce is recorded on the y-axis and the time between bounces is recorded on the x-axis.

For the next part of the lesson, we will select three points from each curve and have the calculator find the equation of a parabola that best fits each set of three points. This process of "curve fitting" is called "regression analysis". In order to complete this lesson, each set of three points has been pre-selected from another actual experiment. In this second experiment, the sequence of bounces was interrupted then resumed:
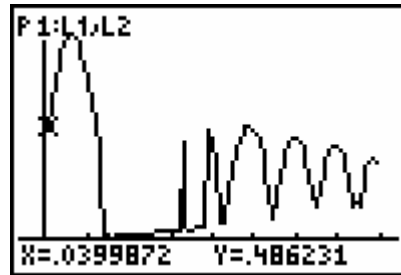
The following values have been pre-selected from this experiment where the path of bouncing ping pong ball was traced using the graphing calculator, a CBL, and a motion detector:

$$X = \{.0399, .320, .560, 2.199, 2.440, 2.599, 2.839, 2.959, 3.079, 3.239, 3.439, 3.639, 3.759, 3.919, 3.959\}$$

$$Y = \{.486, .880, .593, .210, .484, .354, .337, .436, .379, .134, .397, .249, .145, .344, .302\}$$

For this set of data, the x-values represent the time intervals that data was recorded to the nearest $1/1000^{th}$ of a second. The y-values represent the height of the ball from the floor to the nearest $1/1000^{th}$ of a meter.

In order to analyze this data on the calculator, we must first enter the data and have it displayed. To do this, perform the following keystrokes. (To examine the areas of the calculator utilized by these commands in more detail, consult your calculator's handbook.)

Step #1: Press, $\boxed{\text{STAT}}$ and select #4 for, 'ClrList'. Press, $\boxed{\text{2nd}}$, #1, $\boxed{\text{,}}$, $\boxed{\text{2nd}}$, #2, $\boxed{\text{,}}$, $\boxed{\text{2nd}}$, #3, $\boxed{\text{,}}$, $\boxed{\text{2nd}}$, #4, $\boxed{\text{ENTER}}$ (the comma key is above the number 7).

-This clears the calculator of any previous information that may interfere with our data.

Step #2: Press, $\boxed{\text{STAT}}$, #1, and type in the x-values of the selected data listed above in the order that they are listed in each set. Press, $\boxed{\text{ENTER}}$ after each number is typed and be sure that you are typing these numbers into L_1 . Once all

x-values have been entered into L_1 repeat the process for the y-values, entering these coordinates into L_2. Again, be sure to enter these values in the order listed. Do not sort numbers in either list.

Step #3: Press $\boxed{Y =}$ and clear this editor of any previous equations. Press $\boxed{2nd}$, $\boxed{Y =}$ where you will come to the "STAT PLOTS" menu. If any of these indicate to be "on", select #4, "PlotsOff", then press $\boxed{ENTER}$, $\boxed{ENTER}$ to turn all "STAT PLOTS" off.

Step #4: Return to the "STAT PLOTS" menu and select, "Plot1", then press $\boxed{ENTER}$.

> a.) Turn the plot 'on' by selecting "ON", then $\boxed{ENTER}$.
> b.) Use your down arrow to "Type:". Select the first display 'icon' beside 'Type' and press $\boxed{ENTER}$.
> c.) Use your down arrow to, "Xlist:" and type, $\boxed{2nd}$, #1 to use L_1 as your x-values.
> d.) Use your down arrow to, "Ylist:" and type, $\boxed{2nd}$, #2 to use L_2 as your y-values.
> e.) Use your down arrow to, "Mark:", select the first icon (a box) and press $\boxed{ENTER}$.
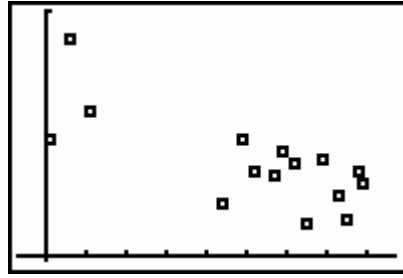
Your calculator should display the following:
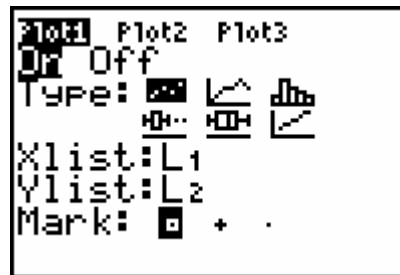
Step #5: Press $\boxed{WINDOW}$ and enter the following values:

> a.) Xmin= -0.35
> b.) Xmax= 4.35
> c.) Ymin= -0.01
> d.) Ymax= 1.00
> e.) Press $\boxed{GRAPH}$

The following should now be displayed:

The above graph is called a "Scatter Plot". In this graph every three displayed points are on one parabola (left to right). We will now use the calculator's "Regression Equation" menu to find an equation of a parabola for every three points.

Step #1: Press $\boxed{\text{STAT}}$, #1 and right arrow to L_3 . Type in the first three x-values into L_3 , then type in the first three y-values into L_4 .

Once the first three x and y-values have been entered into L_3 , L_4 we are ready to have the calculator predict the equation of the parabola that best fits these three points.

*Step #2*: Press $\boxed{\text{STAT}}$ and right arrow to the "CALC" menu.

    a) Select #5, for "QuadReg" (quadratic regression).
    b) With the cursor flashing to the right of this command type, $\boxed{\text{2nd}}$, #3, $\boxed{}$, $\boxed{\text{2nd}}$, #4, to use L_3 and L_4 in the regression equation.
    c) Press $\boxed{\text{ENTER}}$.

The calculator has now found values for a, b, & c, which can be used in the standard equation of a parabola for our first set of three points. Write these numbers down rounded to two or three decimal places.

Step #3: After writing down these values for a, b, & c, press $\boxed{\text{Y} =}$ and enter the equation of the first parabola:

$$Y_1 = -5.004X^2 + 3.208X + 0.366 \text{ and then press } \boxed{\text{GRAPH}} .$$

A parabola is now graphed which contains the first three points of our data. This curve represents the path of our Ping-Pong ball on its first bounce.
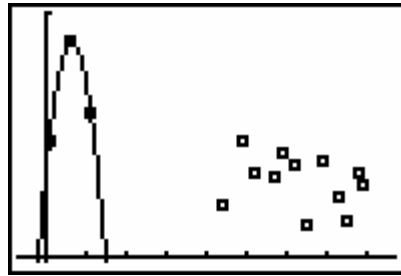
Step #4: Press $\boxed{\text{STAT}}$, #1 and return to L_3 and L_4 . Type in the next three x and y -values into L_3 , L_4 **replacing** the existing numbers.

a.) Repeat step #2 above to find a, b, & c for the second set of three points.

b.) Enter your second equation into $\boxed{Y_2 \ =}$, the third equation into $\boxed{Y_3 \ =}$ and so on.

c.) Repeat this process until you have calculated and graphed a, b, & c values for each remaining set of three points.

Your final graph should appear as the following:
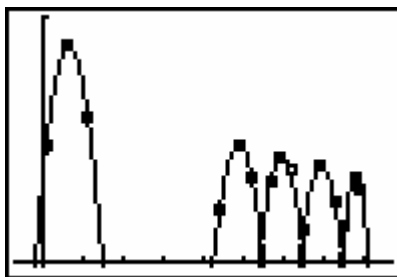
You will notice two important features about this graph:

1. The height of the parabolas steadily declines. The force of gravity overtaking the ping pong ball's bounce caused this decline. Eventually, had the experiment proceeded, the parabolas would diminish to the x-axis.

2. The graph appears to show that one or two parabolas are missing from our data. This is indeed the case. In a real-world situation, data is often collected where complete sets are missing. This can be caused by many reasons: faulty technology, human error, environmental interference, etc. Later in this course - after we have examined the mathematics of exponential and logarithmic functions - we will return to this data and fill in the missing curves, mathematically, in order to "predict" how the ping pong ball should have bounced over the "gap" in our graph.

Completing the Analysis of the Ping-Pong Ball Problem in Unit #13

Part 1: Finding the Exponential Curve to Model Our Data

Recall in Unit #13 that we used a sequence of parabolas to model a bouncing ping pong ball and noticed that, as gravity overcame the ball's bouncing, the height of each parabola declined according to the following graph:
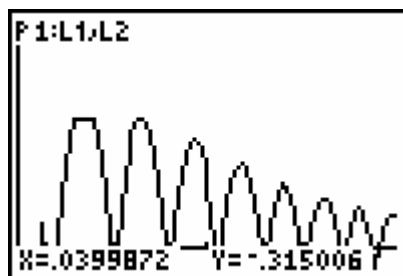
Later in the unit, we described how to use the calculator's regression equation editor to find equations for each parabola based on collected data.

At this time, students should re-enter the data into their calculator's lists and display their results. (For convenience, the original data are listed below (Enter data into L_1 and L_2 and be sure that these are the only numbers in these lists OR see the unit link to "Scatter Plots and Regression Equations (From Unit #13))

$$X = \{.0399, .320, .560, 2.199, 2.440, 2.599, 2.839, 2.959, 3.079, 3.239, 3.439, 3.639, 3.759, 3.919, 3.959\}$$

$$Y = \{.486, .880, .593, .210, .484, .354, .337, .436, .379, .134, .397, .249, .145, .344, .302\}$$

Step #1: To enter the numbers press $\boxed{\text{STAT}}$, $\boxed{\text{ENTER}}$, then type in the numbers. When you are done there should be 15 numbers in L_1 and 15 in L_2. If there are additional numbers in either list, highlight each number then press $\boxed{\text{DEL}}$ to clear that number from the list. It is important that each list contain only the 15 pairs of numbers listed above.

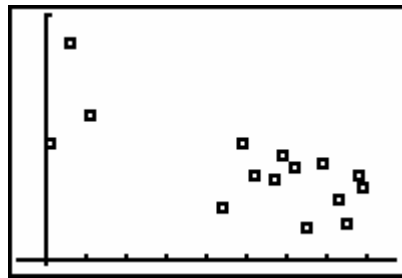*Step #2*: Set your calculator's viewing window to the following

       Xmin = -0.35   Ymin = -0.2

       Xmax = 4.35   Ymax = 1.00

Step #3: Press 2nd, Y = and turn on Plot #1. (Be sure all other Plots are off). Highlight each area of the Plot #1 screen and setup this section so that it appears as follows:

When this is done, press GRAPH to review the original data from **Unit #13**.

Step #4: Recall in Unit #13 that we used the calculator to determine a "Regression Equation" for each set of three points. Finishing this task was left to the student as an exercise for that unit. The steps required to complete this will not be repeated in this unit. The final equations have already been calculated and are written below. Type these equations into Y= exactly as written, and then press GRAPH.
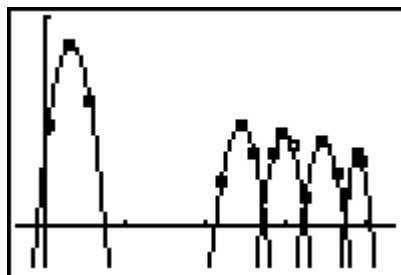
$$Y_1 = -5.004X^2 + 3.208X + 0.366$$

$$Y_2 = -4.886X^2 + 23.8047X - 28.5081$$

$$Y_3 = -5.4167X^2 + 32.2308X - 47.5084$$

$$Y_4 = -5.1375X^2 + 35.6232X - 61.3515$$

$$Y_5 = -11.4688X^2 + 89.3008X - 173.4824$$

(When finished your graph should appear as follows):

We are now ready to complete this exercise by using our knowledge of exponents and logarithms, along with our previous knowledge of parabolas, to fill in the missing data.

Step #5: Press $\boxed{\text{STAT}}$, $\boxed{\text{ENTER}}$ to return to the lists. Scroll right so that L_3 and L_4 come into view.

From our displayed graph it is apparent that the vertex of each parabola is the maximum height obtained by the ball on each bounce. We will model an exponential curve for these vertices so that we can obtain a maximum height for the missing parabolas in our data. In the assignment for this unit, students will be asked to model a logarithmic curve to fit the data and assess which of the two curves is a better choice to model the behavior of a bouncing ball.

The vertex of each parabola is the 2nd point in every set of three data points in L_1 and L_2. To have the calculator use these values to obtain an exponential curve we need to copy these values into L_3 and L_4.

Type the following into L_3, beginning with the first "cell". (Remember that only these values should be typed in L_3 and L_4 and any extra numbers should be deleted).

$L_1(2)$, $\boxed{\text{ENTER}}$

$L_1(5)$, $\boxed{\text{ENTER}}$

$L_1(8)$, $\boxed{\text{ENTER}}$

$L_1(11)$, $\boxed{\text{ENTER}}$

$L_1(14)$, $\boxed{\text{ENTER}}$

Scroll to the first cell in L_4 and type in the following:

$L_2(2)$, $\boxed{\text{ENTER}}$

$L_2(5)$, $\boxed{\text{ENTER}}$

$L_2(8)$, $\boxed{\text{ENTER}}$

$L_2(11)$, $\boxed{\text{ENTER}}$

$L_2(14)$, $\boxed{\text{ENTER}}$

(Be sure to leave these values in your calculator, as they will be needed in the assignment for this unit).

Step #6: We now need the calculator to find an exponential **"curve of best fit"** for the vertices that will serve as the maximum height obtained by the bouncing Ping-Pong ball anywhere along the graph – and therefore in the missing region as well.

Type the following from your home screen:

$\boxed{\text{STAT}}$, Right arrow to "CALC", scroll down to "ExpReg", $\boxed{\text{ENTER}}$.
Now type $\boxed{\text{2nd}}$, L_3, $\boxed{\text{2nd}}$, L_4, $\boxed{\text{ENTER}}$.

The calculator has now calculated the best values for $"a"$ – the initial condition- and $"b"$ - the base – of an exponential function. Record these values to four decimal places, and then enter the following into Y_6 and press $\boxed{\text{GRAPH}}$.

$Y_6 = 0.9433*0.7721\textasciicircum X$

Your calculator should display the following:
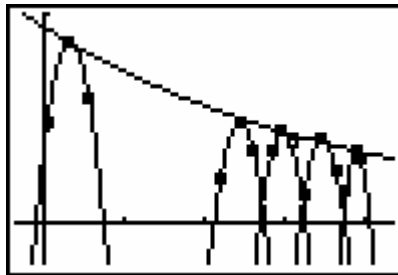
The exponential curve connecting the vertices of the parabolas can now be used as the maximum vertical height that would have been obtained by the bouncing Ping-Pong ball. On this curve will be the vertices of the two missing parabolas. We can locate these vertices using simple algebraic techniques. Then using two other points- found through logical speculation - equations for the missing parabolas can be estimated and used as substitutes for the data that was not recorded.

Part 2: Filling the Gaps of the Missing Parabolas

As you look at your graph, it should appear that two parabolas are missing, or were not collected. Our task is to find equations for these two missing parabolas.

In a real world setting, highly specialized mathematics would be applied to fill in the gaps of collected data. With the advent of computer technology, an enormous amount of numerical data can be processed quickly and accurately. Still the procedures involved rely on the mathematician.

The word **"Algorithm"** is used to describe these sets of procedures – or in the case of computers, they are sets of programming commands. For our purposes, the first part of algorithm was accomplished in Unit #13 when we:

- Entered and displayed our data.
- Used the calculator's regression equation editor to find parabolic equations to fit the data.

In this unit we will complete our algorithm by:

- Finding an exponential equation to serve as the maximum height obtained by the bouncing Ping-Pong ball. (already completed).
- Locate the vertices of the two missing parabolas as points on the exponential curve.
- Locate two other points-logically-that would likely be on the two missing parabolas so that they can be used, with each vertex, to find equations of the missing parabolas.
- Display the data and speculate about whether better estimates for our data are available (unit assignment).

Estimating the Coordinates of the Missing Vertices

For simplicity, we will assume that the missing vertices are equally spaced between the vertices of the displayed parabolas to the left and to the right. This implies that we need to locate x-values that are $\frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ between the gap in our data. To find these values, we will first $\boxed{\text{TRACE}}$ to where the gap begins and ends.

Step #1: Press $\boxed{\text{TRACE}}$ and down arrow until you are tracing along the first parabola (found in Y_1). Trace to a point that is as close to the point of intersection between this parabola and the x-axis as is possible. (Using the technique of "successive approximations", outlined in a previous unit, a better estimate of this point can be made. To save time we will use only the point values provided by the calculator. Students may wish to employ "successive approximations" on their own).
Record the first traced value to four decimal places.

$x = 0.7487$ $\qquad\qquad$ $y = -0.0371$

Step #2: Down arrow so that you are tracing along the first parabola to the right of the gap (found in Y_2) and trace to the closest point of intersection between this parabola and the x-axis.

$x = 2.0995$ $\qquad\qquad$ $y = -0.0671$

Step #3: Find the average of the two x-values recorded. This will locate the point that is one-half way between the endpoints of the data gap.

$$\frac{0.7487 + 2.0995}{2} = 1.4241$$

Record this result as the coordinate point (1.4241, 0). This point will be on both missing parabolas

Step #4: Find the x-values of the vertices of the missing parabolas by finding the value half way between the recorded x-value to the left and value calculated in *step #2*, then use the recorded x-value to the right and the value found in *step #3*.

First missing vertex x-value:

$$\frac{0.7487 + 1.4241}{2} = 1.0864$$

Second missing vertex x-value:

$$\frac{1.4241 + 2.0995}{2} = 1.7618$$

Step #5: Find the y-value of each missing vertex. $\boxed{\text{TRACE}}$ on your calculator until you are tracing on the exponential curve found in Y_6.

First missing vertex y-value:

Type 1.0864 and record $y = 0.7122$

Second missing vertex y-value:

Type 1.7618 and record $y = 0.5980$

Step #6: Display the two vertices and three calculated points along the gap on the x-axis by adding their coordinates to
L_3 and L_4

Press $\boxed{\text{STAT}}$, edit and scroll down to the bottom of L_3 then enter the x-values (in order) of the 5 points recently calculated.

$L_3(6) = 0.7487,$ ENTER (beginning of gap)

$L_3(7) = 1.0864,$ ENTER (first missing vertex)

$L_3(8) = 1.4241,$ ENTER (midpoint of gap)

$L_3(9) = 1.7618,$ ENTER (second missing vertex)

$L_3(10) = 2.0995,$ ENTER (end of gap)

Cursor to the bottom of L_4 and enter the following y-values
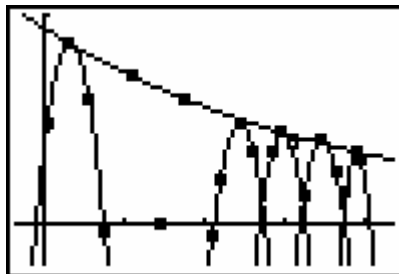
$L_4(6) = -0.0371$

$L_4(7) = 0.7122$

$L_4(8) = 0$

$L_4(9) = 0.5980$

$L_4(10) = -0.0671$

Note : Press 2nd, $Y =$ and turn on Plot #2 for Lists 3 and 4.

Step #7: Press GRAPH to view your results.

Finding the Equations of the Missing Parabolas

Step #1: Press $\boxed{\text{STAT}}$ and scroll right till L_5 and L_6 come into view. Empty any excess numbers in these lists as previously outlined.

We will now have the calculator find a **"Curve of Best Fit"** for each missing parabola. (Recall in Unit #13 that we used both matrices and the calculator to find equations of parabolas. In this unit, only the calculator will be used. The student may wish to review Unit #13's methods for using matrices and consider their use when performing the assignment for this unit).

Step #2: Enter the first three points for the first missing parabola into L_5 and L_6. These points are found in L_3 and L_4 in cells 6 – 8, or they can be entered directly from the student's notes.

Step #3: Once the first three points are in L_5 and L_6 press, $\boxed{\text{STAT}}$, right arrow to, "CALC", scroll down to "#5" and press $\boxed{\text{ENTER}}$. Type $\boxed{\text{2nd}}$, L_5, $\boxed{\text{2nd}}$, L_6, $\boxed{\text{ENTER}}$.

The calculator has now found values for, *a*, *b*, and *c* in the **standard equation of a parabola**. Record these values to four decimal places, and then repeat the process for finding the calculator's equation of the second missing parabola. Record these values to four decimal places by copying the last three points found in L_3 and L_4 to L_5 and L_6.

Step #4: Enter the following equations into Y_7 and Y_8 on your calculator, then $\boxed{\text{GRAPH}}$.

$$Y_7 = -6.4078X^2 + 13.9777X - 6.9103$$

$$Y_8 = -5.5379X^2 + 19.4140X - 16.4163$$

Your final graph should appear as follows