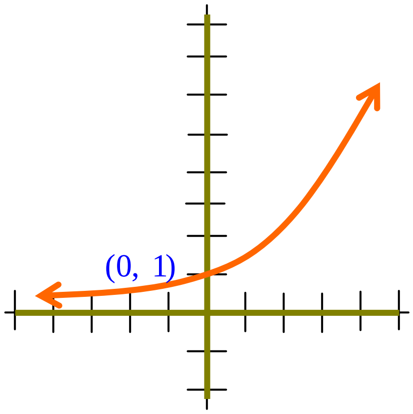
INTRODUCTION TO LOGARITHMS

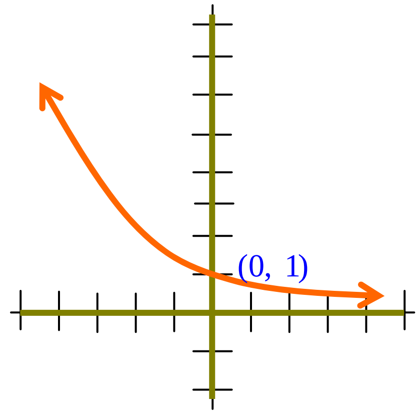
Logarithms are very useful mathematical entities that have an enormous amount of application to real-world situations. Combined with their inverse relationship to the exponential functions, logarithms allow for the simplification of complex problem situations to basic arithmetic operations. In this unit you will examine the definition and inverse relationship with the exponential function, practice the laws of logarithms, solve logarithmic equations, and explore a more efficient method for solving equations using the "Change of Base" formula for logarithms.

Introduction to Logarithms

In the last unit, we discovered that there are four basic graphs of exponential expressions depending on the value of "a". In our studies for this unit, we will only utilize the first two graphs presented and repeated below.



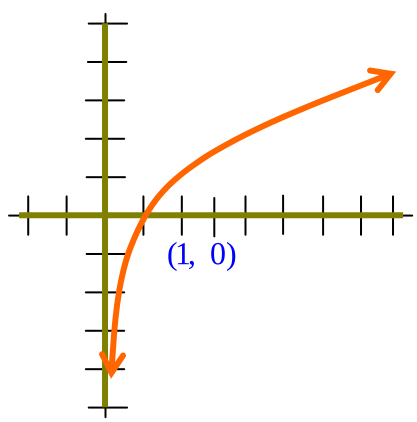
$y = a^x$ for $a > 1$



$y = a^x$ for $0 < a < 1$

Recall that a function is "one-to-one" (1-1) if it passes the "horizontal line test" (HLT). Clearly, both of the above graphs meet the criteria for this test. Also recall that one consequence of a "1-1" function is that its inverse is also "1-1". For the exponential graphs displayed above, the inverse function is called the **"logarithm"** or **"logarithmic curve"** of the exponential function and is displayed below for the inverse of $y = a^x \ : \ a > 1$.

$$f^{-1}(x) = \log_a[f(x)]$$



(Note: It can be shown that the inverse of $y = a^x \ : \ 0 < a < 1$ is logically equivalent to the above graph. Therefore, we will only use the above in our exploration of logarithms).

Definition and Inverse Relationship with the Exponential Function

Logarithms are very useful mathematical entities that have an enormous amount of application to real-world situations. Combined with their inverse relationship to the exponential functions, logarithms allow for the simplification of complex problem situations to basic arithmetic operations which we will demonstrate shortly. Because of their usefulness, the following definition of the logarithmic function is one of the more significant definitions presented in this course.

Logarithmic Function: Given an exponential function of the form, $f(x) = a^x$,

the logarithm function is the inverse function $f^{-1}(x)$ and is defined as:

$$f^{-1}(x) = \log_a[f(x)]$$

where $f^{-1}(x)$ is an exponent on base a, (a^x) whose value equals $f(x)$.

The initial complexity of this definition can be simplified to the following basic and fundamental identity.

$$y = a^x \Leftrightarrow x = \log_a y$$

The "$\Leftrightarrow$" in this identity indicates the inverse relationship that exists between exponential equations and logarithms.

Example #1(A): Rewrite the following exponential equation as a logarithm.

$$5 = 3^x$$

Step #1: Identify the components of the exponential expression.

$$\text{In } 5 = 3^x \ ; \ x = x, \ a = 3, \ y = 5$$

Step #2: Substitute values into the logarithmic function.

$$x = \log_a y \ \Rightarrow \ x = \log_3 5$$

Example #1(B): Rewrite the following exponential equation as a logarithm.

$$15 = (3x)^z$$

Step #1: Identify the components of the exponential expression.

$$a = 3x, \quad x = z, \quad y = 15$$

(Note: $a,\ x,\ y$ in the original inverse relationship should be viewed as 'position placeholders' and not as the actual values in the expression).

Step #2: Substitute values into the logarithmic function.

$$x = \log_a y \quad \Rightarrow \quad z = \log_{3x} 15$$

The key to this transformation is that a variable exponent can be found and assigned a real value. In *Example #1(A)* above $x \approx 1.46497$ (which we will learn to find later). In addition, the value of x can now be manipulated and transformed using ordinary arithmetic (or algebraic) processes.

For example:

$$2x = 2\log_3 5 = 2(1.46497) = 2.9299$$

$$\pi + x = \pi + \log_3 5 = 3.14159 + 1.4649 = 4.60656$$

$$\frac{x^2}{2.85} = \frac{(\log_3 5)^2}{2.85} = \frac{(1.46497)^2}{2.85} = 0.7530341, \qquad \text{and etc.}$$

The fact that x is actually an exponent indicates that we can transform a complex expression involving variable exponents into simple numerical constants, perform basic operations on these constants then replace them as exponents in the original expression to evaluate a given situation. Before we can perform these types of operations, we first need to examine the rules that govern the use of logarithmic quantities.

The Laws of Logarithms

By the inverse relationship:

$$y = a^x \Leftrightarrow x = \log_a y$$

we see that a logarithm is an exponent. Therefore all rules that apply to exponents also apply to logarithms. For example, in the following relationship we let the exponents be replaced by a logarithmic expression.

Recall: $x^m \cdot x^n = x^{m+n}$ Let $m = \log_x y$, $n = \log_x y$

$$\Rightarrow \quad x^{\log_x y} \cdot x^{\log_x y} = x^{2\log_x y}$$

For: $(x^m)^n = x^{mn} \Rightarrow (x^{\log_x y})^{\log_x y} = x^{(\log_x y)^2}$

Note: $(\log_x y)^2 \neq \log_x y^2$

Similar properties and rules can also be established in this manner. The following lists these rules as the **"Laws of Logarithms"**. The students are encouraged to verify any of these properties on their own.

Laws of Logarithms

1.) $\log_a (m \cdot n) = \log_a m + \log_a n$

2.) $\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n$

3.) $\log_a m^p = p \log_a m$

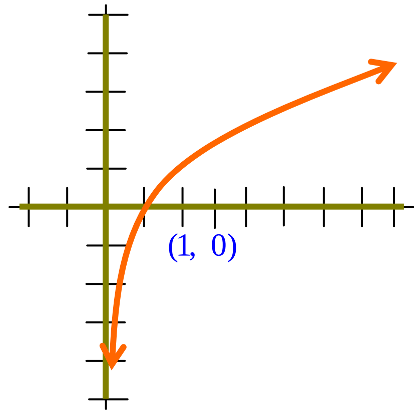
4.) $\log_a m^p \neq (\log_a m)^p$

5.) $\log_a 1 = 0$

6.) $\log_a q = \text{undefined if } q \leq 0$

(The last property can be seen in the graph of the logarithm presented earlier (shown below). Clearly from the graph, the $D_f : \log_a x \; = \; (0, \infty)$, therefore $q \leq 0$ is not defined).

$$f^{-1}(x) = \log_a [f(x)]$$



Solving Logarithmic Equations

It is significant to note that the above Laws apply only to logarithms with identical bases. Just as $x^m \cdot y^n \neq xy^{m+n}$ because of the different bases involved; so too $\log_a x + \log_b y \neq \log_{ab} xy$. For logarithms that involve different bases, we will introduce a "Change of Base" formula at the end of this unit.

Laws of Logarithms

1.) $\log_a (m \cdot n) = \log_a m + \log_a n$

2.) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

3.) $\log_a m^p = p \log_a m$

4.) $\log_a m^p \neq (\log_a m)^p$

5.) $\log_a 1 = 0$

6.) $\log_a q = \text{undefined if } q \leq 0$

Example #1: Solve for *x*: $\log_7 x = \log_7 4$

Clearly, $x = 4$ in this expression. This result is significant as it highlights the usefulness of logarithms as able to equate exponents. When this occurs, the logarithmic notation on the expression is dropped and only the arguments of the equations need be solved –as in the following example.

Example #2: Solve for *x*: $\log_3 (x+2) = \log_3 (3x-4)$

Step # 1: Equate "log" expressions (already done).

Step #2: Solve the equation of arguments.

$$x + 2 = 3x - 4$$
$$x = 3$$

Example #3: Solve for *x*: $\log_a(3x-4)+\log_a 5=\log_a(8x+1)$

Step #1: Equate "log" expressions using "Laws of Logs".

$$\log_a(3x-4)+\log_a 5=\log_a 5(3x-4) \qquad \text{- law \#1}$$

$$\Rightarrow \log_a 5(3x-4)=\log_a(8x+1)$$

Step #2: Solve the equation of arguments.

$$\begin{aligned} 5(3x-4)&=8x+1 \\ 15x-20&=8x+1 \\ x&=3 \end{aligned}$$

Example #4: Solve for *x*: $\log_a(3x+4)+\log_a 5=\log_a(8x-1)$

Using the above *example #4*, we arrive at the following equation of arguments

$$\begin{aligned} 5(3x+4)&=8x-1 \\ x&=-3 \quad \Rightarrow \varnothing \end{aligned}$$

The reason this equation has no solution is that the domain of $f(x)=\log_a x$ is $x>0$ (argument > 0). If we substitute $x=-3$ into either side of the original equation we obtain:

$$\log_a(-9+4)+\log_a 5=\log_a(-24-1)$$

$$\log_a(-5)+\log_a 5=\log_a(-25)$$

Although by 'law #1' we obtain:

$$\log_a(-5\cdot 5)=\log_a(-25) \qquad \text{or} \qquad \log_a(-25)=\log_a(-25)$$

and $-25=-25$

The result is still **invalid**, as at least one expression in the argument is less than 0, which is outside of the domain of logarithms.

Example #5: Solve for x: $\log_5(6-7x)-\log_5(x)=\log_5 38$

Step #1:

$$\log_5(6-7x)-\log_5(x)=\log_5\left(\frac{6-7x}{x}\right) \qquad \text{-law \#2}$$

Step #2: $\log_5\left(\frac{6-7x}{x}\right)=\log_5 38$

$$\frac{6-7x}{x}=38$$

$$6-7x=38x$$

$$\frac{2}{15}=x$$

Step #3: Restrictions

For $\log_5(6-7x) \;\Rightarrow\; 6-7x > 0$

$$x<\frac{6}{7}$$

For $\log_5(x) \;\Rightarrow\; x>0$

$\therefore\ 0 < \frac{2}{15} < \frac{6}{7}$ "true", therefore $x=\frac{2}{15}$

Example #6: Solve for *x*: $\log_2 x + \log_2(x+3) = \log_2 40$

Step #1: $\log_2 x + \log_2(x+3) = \log_2 x(x+3)$

Step #2: $\log_2 x(x+3) = \log_2 40$

$x(x+3) = 40$

$x^2 + 3x = 40$

$x^2 + 3x - 40 = 0$

$(x-8)(x+5) = 0$

$x = 8, -5$

Step #3: $x > 0$ or $x + 3 > 0$

$x > -3$

$x = -5 < -3$; $\Rightarrow x \neq -5$

Therefore $x = 8$

Example #7: Solve for *x*: $\log x^3 - \log x^2 = \log 15$

Step #1: $\log x^3 = 3\log x$ and $\log x^2 = 2\log x$ -law #3

$\Rightarrow \log x^3 - \log x^2 = 3\log x - 2\log x = \log x$

Step #2: $\log x = \log 15$

$x = 15$

Step #3: $x > 0 \Rightarrow x = 15$

(Note: For this example, no base was written on the log expression. When this occurs, the base is understood to be **"10"** and the logarithm is called **"common"** since our usual numeral system is base 10).

Example #8: Solve for *x*: $\log_5(x-2)-\log_5\left(\frac{1}{x+1}\right)=\log_5 10$

Step #1: $\log_5\left(\frac{1}{x+1}\right)=\log_5(x+1)^{-1}=-\log_5(x+1)$

Step #2: $\log_5(x-2)+\log_5(x+1)=\log_5 10$

$$\log_5(x-2)(x+1)=\log_5 10$$

$$(x-2)(x+1)=10$$

$$x^2-x-2=10$$

$$x^2-x-12=0$$

$$(x-4)(x+3)=0$$

$$x=4,\ \ -3$$

Step #3: For $\log_5(x-2)\ \Rightarrow\ x-2>0$ and $x>2$

For $\log_5(x+1)\ \Rightarrow\ x+1>0$ and $x>-1$

Therefore $x\neq-3$ and $x=4$

As noted in the above examples, in order to solve logarithmic equations, logarithmic expressions must first be equated, singly, on both sides of the = sign. In the next examples, we use the inverse property of exponential equations and logarithms to solve the equations.

Example #9: Solve for *x*: $\log_5(x-4)=2$

Step #1: From $y=a^x \Leftrightarrow x=\log_a y$, identify $a,\ x,\ y$

$a=5$ (Base)

$x=2$ (Exponent)

$y=x-4$ (Argument Value)

Step #2: Transform according to inverse relationship, and then solve.

$$\log_5(x-4)=2 \quad\Rightarrow\quad \begin{aligned} 5^2&=x-4\\ 25&=x-4\\ 29&=x \end{aligned}$$

Example #10: Solve for *x*: $\log_2 x-\log_2(x-1)=3$

Step #1: $\log_2 x-\log_2(x-1)=\log_2\dfrac{x}{x-1}$ -law #2

$a=2$ (Base)

$x=3$ (Exponent)

$y=\dfrac{x}{x-1}$ (Argument Value)

Step #2: Transform by inverse relationship:

$$\log_2\left(\frac{x}{x-1}\right)=3 \quad\Rightarrow\quad 2^3=\frac{x}{x-1}$$

$$8=\frac{x}{x-1}$$

$$8x-8=x$$

$$x=\frac{8}{7}$$

Step #3: Restrict $x > 0$ and $x \neq 1$ and $x - 1 > 0 \Rightarrow x > 1$

Therefore $x = \frac{8}{7}$

Example #11: Solve for *x*: $\log_5 13 = x$

Step #1: $5^x = 13$

$x = \log_5 13$

$x = ?$

This last example does have a real solution that we could estimate by graphing $y = 5^x$ as we did in the last unit. But now that we are familiar with manipulating logarithms, we will introduce a more efficient method for finding x by introducing the, **"Change of Base"** formula. See the unit link to "Change of Base Formula for Logarithms".

Change of Base Formula for Logarithms

$$\log_a n = \frac{\log_b n}{\log_b a}$$

As noted earlier, the "common" base of a logarithm is base 10. If we allow $b = 10$ in the above formula, we can evaluate the problem below (Example #11 from the unit link to "Solving Logarithmic Equations") with greater precision as the value of these **"common logarithms"** are already programmed into the graphing calculator.

Example #1: (This example is a continuation of *Example #11* from the unit link to "Solving Logarithmic Equations")

Solve for *x*: $\log_5 13 = x$

$$5^x = 13$$

$$x = \log_5 13$$

$$x = ?$$

Step #1: $\log_5 13 = \frac{\log 13}{\log 5}$ by change of base.

Step #2: Enter the following into your calculator.

[LOG], 13, [÷], [LOG], 5, [ENTER]

$$x = 1.59369$$

which is easily verified as

$$5^{1.59369} = 12.9999474 \approx 13$$

Example #2: Solve for x": $\log_5 29 = x$

Step #1: $\log_5 29 = \dfrac{\log 29}{\log 5}$ (change of base)

Step #2: Use calculator: $\log(29)/\log(5) = 2.09222 = x$

Miscellaneous Examples

Example #1: Solve for x: $\log_x 5 = 9$

Step #1: $\log_x 5 = 9 \quad \Rightarrow \quad x^9 = 5 \quad \Rightarrow \quad x = ?$

Step #2: Use change of base formula.

$$\frac{\log 5}{\log x} = 9$$

$$\Rightarrow \quad \frac{\log 5}{9} = \log x \approx 0.0776633$$

$$x \approx 10^{0.0776633} \approx 1.19581$$

You could enter the following into your calculator:

[2ND][LOG], [LOG], 5, [÷], 9, [ENTER]

Example #2: Solve for x: $5^x = 4^{x+3}$.

Since $\log_a p = \log_a q \quad \Rightarrow \quad p = q$

We can rewrite or, "raise to a log", the given expression as

$$\log 5^x = \log 4^{x+3}$$

$$\Rightarrow \quad x \log 5 = (x+3)\log 4$$

$$x \log 5 = x \log 4 + 3 \log 4$$

$$x(\log 5 - \log 4) = 3 \log 4$$

$$x = \frac{3 \log 4}{\log 5 - \log 4} \approx 18.637705$$

Example #3: Solve $8^{3x-1} = 30.4$

$$\Rightarrow \qquad 3x - 1 = \log_8 30.4$$

$$3x - 1 = \frac{\log 30.4}{\log 8}$$

$$3x - 1 = 1.642$$

$$3x = 2.642$$

$$x = 0.88067$$