TRIGONOMETRY AND THE COMPLEX PLANE

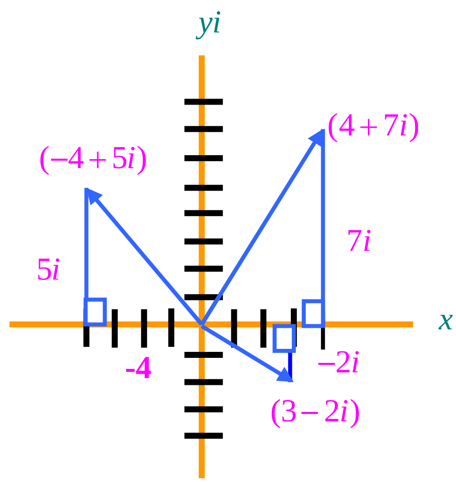
In previous units, complex numbers were introduced and explored, and the "ArcTangent" was used to find to rotation angles when two complex numbers were multiplied. The complex plane and trigonometry are intricately related and have many applications in modern technology. Computer circuitry and electronics are highly dependent on the interconnections between trigonometry and complex numbers. In the branch of mathematics known as "Analytic Number Theory", complex numbers, trigonometry, and logarithms may hold the key to discerning a pattern for the distribution of prime numbers. Since high-level security systems depend on the prime factors of composite integers, a method for deciding the distribution of prime numbers would be an invaluable tool for government and private use. In this unit we will explore the relationships between trigonometry and complex numbers, develop methods for converting complex numbers to trigonometric form (and vice versa), and we will find roots of complex numbers using trigonometric techniques.

Trigonometry on the Complex Plane

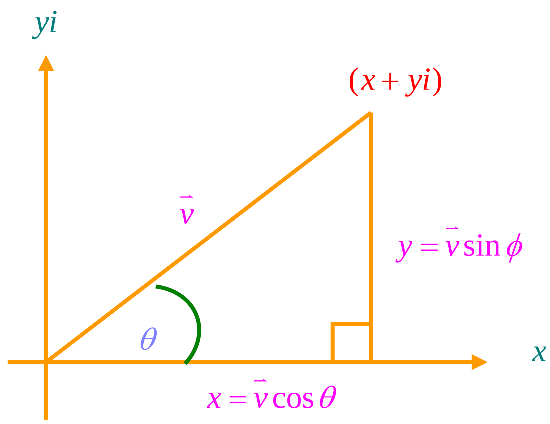
In previous units, the six trigonometric ratios were defined by examining the ratios between the legs and the hypotenuse of a right triangle. We examined various graphing techniques for characteristic curves of trigonometric ratios. The coordinates of the points of these characteristic curves were found by inscribing a right triangle inside the unit circle on the xy-plane. We established and used trigonometric and coordinate identities in translating polar and rectangular coordinates. Until now our studies of these trigonometric relationships occurred primarily on the real number xy-plane. We now extend our knowledge of right triangle trigonometry to the **complex plane**.

Recall that the standard form of a complex number is $a+bi$. Also recall that a complex number can be located on the complex plane by allowing a to represent the horizontal or x component while bi represents the vertical or y component. Once the complex number's location is found on the plane, we then complete a right triangle by drawing the number's **vector** from the origin. The following diagram displays this process:

Graph: $4+7i$ $3-2i$ $-4+5i$



If we account for the acute angles of each triangle whose vertex is at the origin, we can evaluate trigonometric ratios for every complex number. In the diagram below, we will develop a complex number's coordinate identity. Although only a triangle shown in Quadrant #1 is labeled, the derived identity is valid for any value of $\theta \in [0, 2\pi]$.



From this diagram we have:

$$\cos\theta = \frac{x}{\vec{v}} \qquad \sin\theta = \frac{y}{\vec{v}}$$

$$\Rightarrow$$

$$\vec{v}\cos\theta = x \qquad \vec{v}\sin\theta = y$$

$$\Rightarrow x + yi = \vec{v}\cos\theta + \vec{v}\sin\theta(i)$$

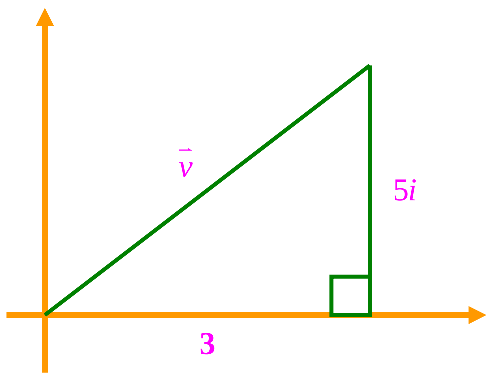
$$\boxed{x + yi = \vec{v}(\cos\theta + i\sin\theta)}$$

which is the coordinate identity for any complex number.

***Note:** In math it is common practice to designate a small case letter "z", such as let $z = 4 - 7i$.

Example #1: Convert the complex number $3+5i$ to polar (trigonometric) form.

Step #1: Find $\left|\vec{v}\right|$. (Recall that $\left|\vec{v}\right|$ is the magnitude of the vector of the complex number). Since this quantity represents the hypotenuse of a right triangle on the complex plane, we use the Pythagorean Theorem to find this quantity.



$$\left|\vec{v}\right|^2 = 3^2 + 5^2$$

$$\left|\vec{v}\right|^2 = 34$$

$$\left|\vec{v}\right| = \sqrt{34}$$

Step #2: For the complex number $3+5i$

$$x = 3 \qquad\qquad yi = 5i$$

Using the complex coordinate identity found above we have

$$3 = \sqrt{34}\cos\theta \qquad\qquad 5i = i\sqrt{34}\sin\theta$$

Therefore: $\sqrt{34}\cos\theta + i\sqrt{34}\sin\theta$ is the polar form of the complex number $3+5i$ for appropriate values of θ.

Example #2: Find the complex polar coordinates of $-2+3i$ and establish values for θ, where $\theta \in [0, 2\pi]$: Set your calculator to degree mode.

Step #1: Find $|\vec{v}|$ and translate components to complex polar identity.

$$|\vec{v}|^2 = (-2)^2 + 3^2$$

$$|\vec{v}|^2 = 13$$

$$\vec{v} = \sqrt{13}$$

$$\Rightarrow -2 = \sqrt{13}\cos\theta \qquad \textit{and} \qquad 3i = i\sqrt{13}\sin\theta$$

Therefore: $-2+3i = \sqrt{13}\cos\theta + i\sqrt{13}\sin\theta$

Step #2: Find appropriate values of θ.

$$-2 = \sqrt{13}\cos\theta \qquad \Rightarrow \qquad \cos\theta = \frac{-2}{\sqrt{13}}$$

and

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{13}}\right)$$

which can be found using your calculator as $\theta \approx 124°$.

also

$$3i = i\sqrt{13}\sin\theta \quad \Rightarrow \quad \frac{3}{\sqrt{13}} = \sin\theta$$

and

$$\theta = \sin^{-1}\left(\frac{3}{\sqrt{13}}\right) \qquad\qquad \theta \approx 56°$$

Step #3: Choose the appropriate value for θ and write $-2+3i$ in complete complex polar form.

If graphed $-2+3i$ is plotted in $Q\#2$ of the complex plane. For $\theta = 124°$ or $56°$, only $\theta = 124°$ is a $Q\#2$ value; therefore, choose $\theta = 124°$.

Hence: $-2+3i = \sqrt{13}(\cos 124° + i\sin 124°)$

Example #3: Convert: $z = 3(\cos 210° + i\sin 210°)$ to rectangular complex form.

Solution: $210° = \dfrac{7\pi}{6}$ $\Rightarrow$ $z = 3\left(\cos\dfrac{7\pi}{6} + i\sin\dfrac{7\pi}{6}\right)$

$$z = 3\left(\frac{-\sqrt{3}}{2} + i\frac{-1}{2}\right)$$

$$z = \frac{-3\sqrt{3}}{2} - \frac{i}{2}$$

Example #4: Convert $z = 2\left(\cos\dfrac{\pi}{18} + i\sin\dfrac{\pi}{18}\right)$ to rectangular complex form.

Solution: Set calculator to radian mode and evaluate:

$$\cos\frac{\pi}{18} \approx 0.985$$

$$\sin\frac{\pi}{18} \approx 0.174$$

$$\Rightarrow \quad z = 2(0.985 + 0.174i)$$

$$\Rightarrow \quad z = 1.970 + 0.347i$$

Complex Powers and DeMoivre's Theorem

In a previous unit, complex numbers such as $7+3i$ and $4-2i$ were multiplied according to the common process, "FOIL".

$$\begin{aligned}(4-2i)(7+3i) &= 28+12i-14i-6i^2 \\ &= 28-2i+6 \\ &= 34-2i\end{aligned}$$

This process can also be used to multiply complex numbers written in polar form; but, the process consumes time and space.

If, for instance, you are asked to

multiply $z=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ by itself 6 times

or in other words

find $z^6=\left[\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]^6$, the process would extend to several pages.

Fortunately a process for multiplying complex polar coordinates was proved by Abraham DeMoivre in 1730. DeMoivre's proof greatly simplifies the process for raising a complex Polar Number to a power or for finding "n^p" complex roots of a complex number.

DeMoivre's Theorem

Let $z = \vec{v}(\cos\theta + i\sin\theta)$ be a complex polar number.

Then $z^n = (\vec{v})^n(\cos(n\theta) + i\sin(n\theta)):\quad n \in \mathbb{N}$

The proof of this theorem, though not difficult, is beyond the current topic and the student may wish to research the process on his/her own.

Example #1: Find $(3+4i)^4$ in rectangular form:

Step #1: Convert $3+4i$ to polar form

$$\left|\vec{v}\right|^2 = 3^2 + 4^2 = 25$$

$$\Rightarrow \left|\vec{v}\right| = 5$$

$$3 = 5\cos\theta \qquad\qquad 4i = 5i\sin\theta$$

$$\frac{3}{5} = \cos\theta \qquad\qquad \frac{4}{5} = \sin\theta$$

$$\theta = \cos^{-1}\left(\frac{3}{5}\right) \qquad\qquad \theta = \sin^{-1}\left(\frac{4}{5}\right)$$

$$\theta \approx 53°$$

(Note: By this time the student should recognize when the calculator should be in either degree or radian mode).

Therefore: $3 + 4i = 5(\cos 53 + i\sin 53)$

Step #2: Substitute values into Demoivre's Theorem

For $(3+4i)^4:\quad n = 4$

$$\Rightarrow \qquad z^n = (\vec{v})^n(\cos(n\theta) + i\sin(n\theta)):\quad n \in \mathbb{N}$$

$$\Rightarrow \qquad z^4 = 5^4(\cos(4\times 53°) + i\sin(4\times 53°))$$

$$z^4 = 625(\cos(212^\circ) + i\sin(212^\circ))$$

$$\approx 625(-0.848 - 0.530i)$$

$$\approx -530 - 331i$$

Example #2: Find $z^6 = \left[\sqrt{3}\left(\cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18}\right)\right]^6$

For this problem, the polar form of the complex number is given. Therefore we identify:

$n = 6$ and use DeMoivre's Theorem.

$$z^6 = (\sqrt{3})^6\left(\cos\left(6\times\frac{5\pi}{18}\right) + i\sin\left(6\times\frac{5\pi}{18}\right)\right)$$

$$z^6 = 27\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

$$z^6 = 27\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$z^6 = \frac{27}{2} - \frac{27\sqrt{3}}{2}i$$

Finding Complex Roots of Complex Numbers

A variation on DeMoivre's Theorem allows us to evaluate such expressions as:

$$\sqrt[4]{(2-7i)} = (2-7i)^{\frac{1}{4}}$$

The formula that results from this variation is:

$$z_k = \sqrt[n]{\vec{v}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \quad \text{where } k = 0,1,2,3.... \quad : \quad n \geq 2$$

Although this formula looks intimidating, its application is not too difficult.

Example #1: Find the complex fourth root of: $z = 4-4\sqrt{3}i$

Step #1: Convert z to polar form:

$$|\vec{v}| = 4^2 + (4\sqrt{3})^2$$

$$|\vec{v}|^2 = 16+48$$

$$|\vec{v}|^2 = 64$$

$$|\vec{v}| = \sqrt{64} = 8$$

$$4 = 8\cos\theta \qquad -4\sqrt{3} = 8\sin\theta$$

$$\frac{1}{2} = \cos\theta \qquad \frac{-\sqrt{3}}{2} = \sin\theta$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) \qquad \theta = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

$$\theta = \frac{\pi}{3} \qquad \theta = \frac{4\pi}{3}, \frac{5\pi}{3}$$

If graphed $4-4\sqrt{3}i$ is plotted in $Q\#4$ of the complex plane.

For $\theta=\frac{\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$, only $\theta=\frac{5\pi}{3}$ is a $Q\#4$ value; therefore, choose $\theta=\frac{5\pi}{3}$.

Therefore: $z=8\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$

Step #2: $n=4$: $k=0,1,2,3 \Rightarrow$ values for the first four roots of z

$\Rightarrow k=0$

$$z_k=\sqrt[n]{v}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right]$$

$$(z)^{\frac{1}{4}}=\sqrt[4]{8}\left(\cos\left(\frac{5\pi}{12}+\frac{2\cdot 0\cdot \pi}{4}\right)+i\sin\left(\frac{5\pi}{12}+\frac{2\cdot 0\cdot \pi}{4}\right)\right)$$

$$=\sqrt[4]{8}\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$$

$\Rightarrow k=1$

$$(z)^{\frac{1}{4}}=\sqrt[4]{8}\left(\cos\left(\frac{5\pi}{12}+\frac{2\pi}{4}\right)+i\sin\left(\frac{5\pi}{12}+\frac{2\pi}{4}\right)\right)$$

$$=\sqrt[4]{8}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$$

$\Rightarrow k=2$

$$(z)^{\frac{1}{4}}=\sqrt[4]{8}\left(\cos\left(\frac{5\pi}{12}+\frac{4\pi}{4}\right)+i\sin\left(\frac{5\pi}{12}+\frac{4\pi}{4}\right)\right)$$

$$=\sqrt[4]{8}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)$$

$\Rightarrow k = 3$

$$(z)^{\frac{1}{4}} = \sqrt[4]{8}\left(\cos\left(\frac{5\pi}{12} + \frac{6\pi}{4}\right) + i\sin\left(\frac{5\pi}{12} + \frac{6\pi}{4}\right)\right)$$

$$= \sqrt[4]{8}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$$

Example #2: Find $(5-i)^{\frac{1}{3}} = z$

Step #1: Convert to Polar form.

$$\left|\vec{v}\right|^2 = 5^2 + (-1)^2 = 26$$

$$\left|\vec{v}\right| = \sqrt{26}$$

$$5 = \sqrt{26}\cos\theta \qquad -i = i\sqrt{26}\sin\theta$$

$$\frac{5}{\sqrt{26}} = \cos\theta \qquad -\frac{1}{\sqrt{26}} = \sin\theta$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{26}}\right) \qquad \theta = \sin^{-1}\left(-\frac{1}{\sqrt{26}}\right)$$

$$\theta \approx 11^\circ\ Q\#1 \qquad \theta \approx -11^\circ\ (Q\#4)$$

Therefore: $z = \sqrt{26}(\cos(-11^\circ) + i\sin(-11^\circ))$

Step #2: $n = 3 \quad k = 0,1,2 \Rightarrow$ values for the first three roots of z

$k = 0$

$$z^{\frac{1}{3}} = (\sqrt{26})^{\frac{1}{3}}\left(\cos\left(\frac{-11}{3} + \frac{2\cdot 0\cdot \pi}{3}\right) + i\sin\left(\frac{-11}{3} + \frac{2\cdot 0\cdot \pi}{3}\right)\right)$$

$$\sqrt[6]{26}\left(\cos\frac{-11^\circ}{3}+i\sin\frac{-11^\circ}{3}\right)$$

$k=1$

$$z^{\frac{1}{3}}=(\sqrt[6]{26})\left(\cos\left(\frac{-11}{3}+\frac{360}{3}\right)+i\sin\left(\frac{-11}{3}+\frac{360}{3}\right)\right)$$

$$\sqrt[6]{26}\left(\cos\frac{349^\circ}{3}+i\sin\frac{349^\circ}{3}\right)$$

$k=2$

$$z^{\frac{1}{3}}=(\sqrt[6]{26})\left(\cos\left(\frac{-11}{3}+\frac{720}{3}\right)+i\sin\left(\frac{-11}{3}+\frac{720}{3}\right)\right)$$

$$\sqrt[6]{26}\left(\cos\frac{709^\circ}{3}+i\sin\frac{709^\circ}{3}\right)$$

Table of Sines, Cosines, and Tangents

Degrees	Radians	cos	sin	tan
0	0	1	0	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	0	1	undefined
120	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
150	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
180	π	-1	0	0
210	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	0	-1	undefined
300	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
330	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
360	2π	1	0	0