De Moivre's formula

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De Moivre's formula, named after Abraham de Moivre, states that for any complex number (and, in particular, for any real number) *x* and any integer *n*,

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx).$$

The formula is important because it connects complex numbers (*i* stands for the imaginary unit) and trigonometry. The expression "cos *x* + *i* sin *x*" is sometimes abbreviated to "cis *x*".

By expanding the left hand side and then comparing the real and imaginary parts, it is possible to derive useful expressions for cos(*nx*) and sin(*nx*) in terms of cos(*x*) and sin(*x*). Furthermore, one can use this formula to find explicit expressions for the *n*-th roots of unity, that is, complex numbers *z* such that $z^n = 1$.

Abraham de Moivre was a good friend of Newton; in 1698 he wrote that the formula had been known to Newton as early as 1676. It can be easily derived from (but historically preceded) Euler's formula $e^{ix} = \cos x + i \sin x$ and the exponential law $(e^{ix})^n = e^{inx}$ (see exponential function).

De Moivre's formula is actually true in a more general setting than stated above: if *z* and *w* are complex numbers, then $(\cos z + i \sin z)^w$ is a multivalued function while cos (*wz*) + *i* sin (*wz*) is not, and one can state that

cos (*wz*) + *i* sin (*wz*) **is one value of** $(\cos z + i \sin z)^w$.

Applications

This formula can be used to find the n^{th} roots of a complex number. If *z* is a complex number, written in polar form as

$$z = A(\cos x + i\sin x),$$

then

$$z^{1/n} = (A(\cos x + i\sin x))^{1/n} = A^{1/n}\left\{\cos\left(\frac{x+2k\pi}{n}\right) + i\sin\left(\frac{x+2k\pi}{n}\right)\right\}$$

where *k* varies from 0 to *n* − 1 to give the *n* roots of the complex number.

Proof by induction

We consider three cases.

For *n* > 0, we proceed by mathematical induction. When *n* = 1, the result is clearly true. For our hypothesis, we assume the result is true for some positive integer *k*. That is, we assume

$$(\cos x + i\sin x)^k = \cos(kx) + i\sin(kx).$$

Now, considering the case $n = k + 1$:

$$\begin{aligned}
&(\cos x + i \sin x)^{k+1} \\
&= (\cos x + i \sin x)^{k} (\cos x + i \sin x) \\
&= (\cos(kx) + i \sin(kx))(\cos x + i \sin x) \text{ (by the induction hypothesis)} \\
&= \cos(kx) \cos x - \sin(kx) \sin x + i(\cos(kx) \sin x + \sin(kx) \cos x) \\
&= \cos(k+1)x + i \sin(k+1)x \text{ , by using trigonometric identities}
\end{aligned}$$

We deduce that the result is true for $n = k + 1$ when it is true for $n = k$. By the Principle of Mathematical Induction it follows that the result is true for all positive integers n.

When $n = 0$ the formula is true since $\cos(0x) + i\sin(0x) = 1 + i0 = 1$, and (by convention) $z^0 = 1$.

When $n < 0$, we consider a positive integer m such that $n = -m$. So

$$(\cos x + i \sin x)^{n} = (\cos x + i \sin x)^{-m}$$

$$= \frac{1}{(\cos x + i \sin x)^{m}} = \frac{1}{(\cos mx + i \sin mx)}\text{, from above}$$

$$\begin{aligned}
&= \cos(mx) - i \sin(mx)\text{, rationalizing the denominator} \\
&= \cos(-mx) + i \sin(-mx) = \cos(nx) + i \sin(nx).
\end{aligned}$$

Hence, the theorem is true for all integral values of n. **Q.E.D.**

See also

- Euler's formula
- Root of unity

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