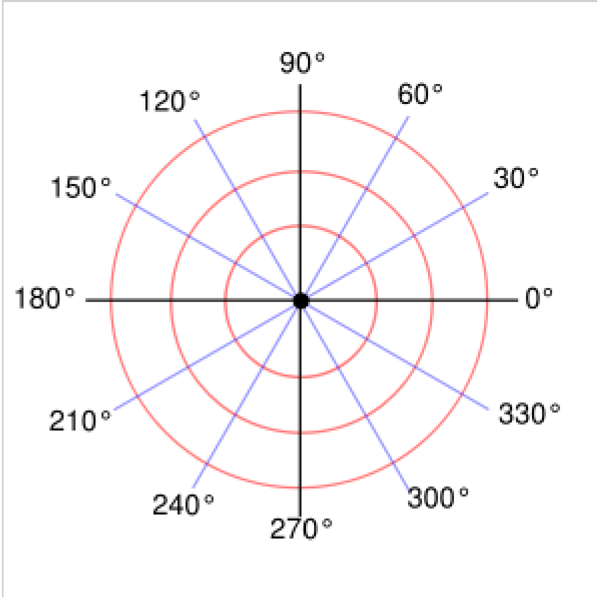
Polar coordinate system

From Wikipedia, the free encyclopedia
(Redirected from Polar coordinates)

In mathematics, the **polar coordinate system** is a two-dimensional coordinate system in which points are given by an angle and a distance from a central point known as the pole (equivalent to the origin in the more familiar Cartesian coordinate system). The polar coordinate system is used in many fields, including mathematics, physics, engineering, navigation and robotics. It is especially useful in situations where the relationship between two points is most easily expressed in terms of angles and distance; in the Cartesian coordinate system, such a relationship can only be found through trigonometric formulae. For many types of curves, a polar equation is the simplest means of representation; for some others, it is the only such means.



A polar grid with several angles labeled.

Contents

History

See also History of trigonometric functions

It is known that the Greeks used the concepts of angle and radius. The astronomer Hipparchus (190-120 BC) tabulated a table of chord functions giving the length of the chord for each angle, and there are references to his using polar coordinates in establishing stellar positions.[1] In *On Spirals,* Archimedes describes his famous spiral, a function whose radius depends on the angle. The Greek work, however, did not extend to a full coordinate system.

There are various accounts of who first introduced polar coordinates as part of a formal coordinate system. The full history of the subject is described in Harvard professor Julian Lowell Coolidge's *Origin of Polar Coordinates.*[2][3] Grégoire de Saint-Vincent and Bonaventura Cavalieri independently introduced the concepts at about the same time. Saint-Vincent wrote about them privately in 1625 and published in 1647, while Cavalieri published in 1635 with a corrected version appearing in 1653. Cavalieri first utilized polar coordinates to solve a problem relating to the area within an Archimedean spiral. Blaise Pascal subsequently used polar coordinates to calculate the length of parabolic arcs.

In *Method of Fluxions* (written 1671, published 1736), Sir Isaac Newton was the first to look upon polar coordinates as a method of locating any point in the plane. Newton examined the transformations between polar coordinates and nine other coordinate systems. In *Acta eruditorum* (1691), Jacob Bernoulli used a system with a point on a line, called the *pole* and *polar axis* respectively. Coordinates were specified by the distance from the pole and the angle from the *polar axis*. Bernoulli's work extended to finding the radius of curvature of curves expressed in these coordinates.

The actual term *polar coordinates* has been attributed to Gregorio Fontana and was used by 18th century Italian writers. The term appeared in English in George Peacock's 1816 translation of Lacroix's *Differential and Integral*

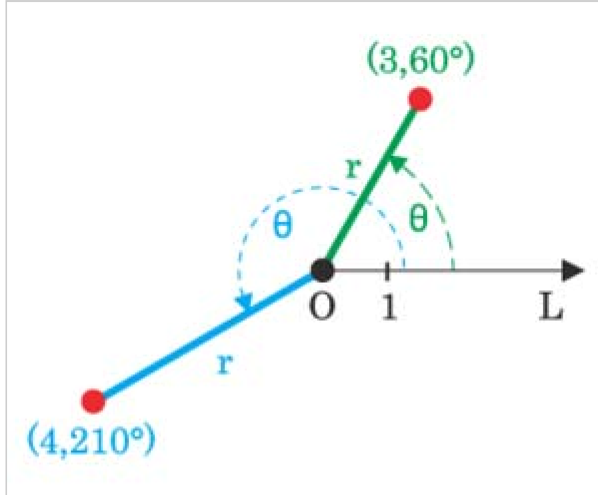
Calculus.[4][5][6]

Alexis Clairaut and Leonhard Euler are credited with extending the concept of polar coordinates to three dimensions.

Plotting points with polar coordinates

As with all two-dimensional coordinate systems, there are two polar coordinates: r (the radial coordinate) and θ (the angular coordinate, polar angle, or azimuth angle, sometimes represented as φ or t). The r coordinate represents the radial distance from the pole, and the θ coordinate represents the anticlockwise (counterclockwise) angle from the 0° ray (sometimes called the polar axis), known as the positive x-axis on the Cartesian coordinate plane.[7]

For example, the polar coordinates (3,60°) would be plotted as a point 3 units from the pole on the 60° ray. The coordinates (−3,240°) would also be plotted at this point because a negative radial distance is measured as a positive distance on the opposite ray (240° − 180° = 60°).



The points (3,60°) and (4,210°)

One important aspect of the polar coordinate system not present in the Cartesian coordinate system is the ability to express a single point with an infinite number of different coordinates. In general, the point (r, θ) can be represented as (r, θ ± n×360°) or (−r, θ ± (2n + 1)180°), where n is any integer.[8] If the r coordinate of a point is 0, then regardless of the θ coordinate, the point will be located at the pole.

Use of radian measure

Angles in polar notation are generally expressed in either degrees or radians, using the conversion 2π rad = 360°. The choice depends largely on the context. Navigation applications use degree measure, while some physics applications (specifically rotational mechanics) use radian measure, based on the ratio of the radius of the circle to its circumference.[9]

Converting between polar and Cartesian coordinates

The two polar coordinates r and θ can be converted to Cartesian coordinates by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

From those two formulas, conversion formulas in terms of x and y are derived, including

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan\frac{y}{x} \qquad x \neq 0$$

[10]If $x = 0$, then if y is positive $\theta = 90°$ ($\pi/2$ radians) and if y is negative $\theta = 270°$ ($3\pi/2$ radians).

Polar equations

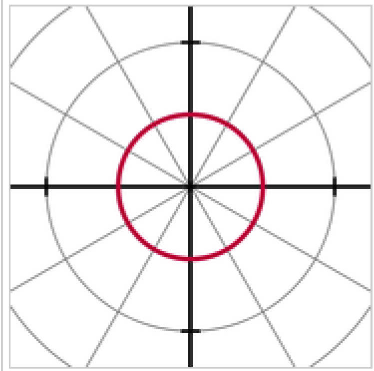
The equation of a curve expressed in polar coordinates is known as a *polar equation*, and is usually written with r as a function of θ.

Polar equations may exhibit different forms of symmetry. If $r(-\theta) = r(\theta)$ then the curve will be symmetrical about the horizontal (0°/180°) ray; if $r(\pi-\theta) = r(\theta)$ then it will be symmetric about the vertical (90°/270°); if $r(\theta-\alpha) = r(\theta)$ then it will be rotationally symmetric α° counterclockwise about the pole.[10]

Circle

The general equation for any circle with a center at (r_0, φ) and radius a is

$$r^2 - 2rr_0\cos(\theta - \varphi) + r_0^2 = a^2$$



A circle with equation $r(\theta) = 1$.

This can be simplified in various ways, to conform to more specific cases, such as the equation

$$r(\theta) = a$$

for a circle with a center at the pole and radius a.[11]

Line

Radial lines (those which run through the pole) are represented by the equation

$$\theta = \varphi,$$

where φ is the angle of elevation of the line; that is $\varphi = \arctan m$ with m the slope of the line in the Cartesian coordinate system.

Any line which does not run through the pole is perpendicular to some radial line.[12] The line which crosses the line $\theta = \varphi$ perpendicularly at the point (r_0, φ) has equation

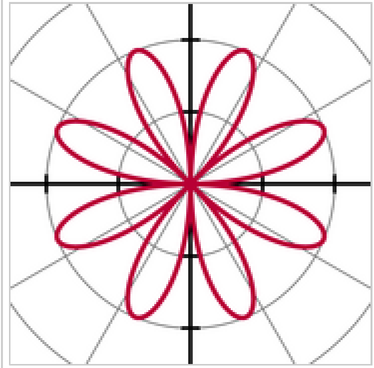
$$r(\theta) = r_0\sec(\theta - \varphi).$$

Polar Rose

A polar rose is a famous mathematical curve which looks like a petalled flower, and which can only be expressed as a polar equation. It is given by the equations

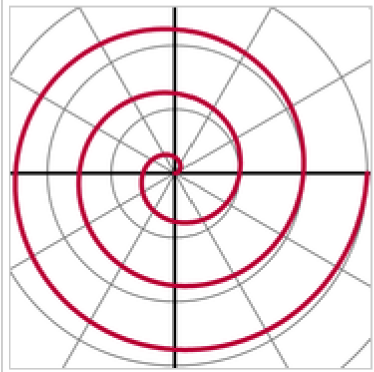
$$r(\theta) = a\cos k\theta \text{ OR}$$
$$r(\theta) = a\sin k\theta$$

If k is an integer, these equations will produce a k-petalled rose if k is odd, or a 2k-petalled rose if k is even. If k is not an integer, a disc is formed, as the number of petals is also not an integer. Note that with these equations it is impossible to make a rose with 2 more than a multiple of 4 (2, 6, 10, etc.) petals. The variable a represents the length of the petals of the rose.



A polar rose with equation $r(\theta) = 2 \sin 4\theta$.

Archimedean spiral



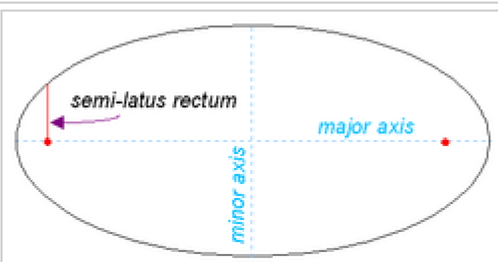
One arm of an Archimedean spiral with equation $r(\theta) = \theta$ for $0 < \theta < 6\pi$.

The Archimedean spiral is a famous spiral that was discovered by Archimedes, which also can be expressed only as a polar equation. It is represented by the equation:

$$r(\theta) = a + b\theta.$$

Changing the parameter a will turn the spiral, while b controls the distance between the arms, which is always constant.. The Archimedean spiral has two arms, one for $\theta > 0$ and one for $\theta < 0$. The two arms are smoothly connected at the pole. Taking the mirror image of one arm across the 90°/270° line will yield the other arm.

Conic sections



Ellipse, showing semi-latus rectum

A Conic section with one focus on the orgin, and the other somewhere on the 0° ray (i.e. the major axis lies along the polar axis) is given by:

$$r = \frac{l}{(1 + e \cos\theta)}$$

where e is the eccentricity and l is the semi-latus rectum, the perpendicular distance at a focus from the major axis to the curve. If $e > 1$ it defines a hyperbola; if $e = 1$ it defines a parabola; and if $e < 1$ this equation defines an ellipse. The special case $e = 0$ of the latter results in a circle of radius l.

Other curves

Because of the circular nature of the polar coordinate system, it is much simpler to describe many curves with an equation in polar rather than Cartesian form. Among these curves are lemniscates, limaçons, and cardioids.

Complex numbers

Complex numbers, written in rectangular form as $a + bi$, can also be expressed in polar form in two different ways:

1. $r(\cos\theta + i\sin\theta)$, abbreviated $r \text{ cis } \theta$
2. $re^{i\theta}$

which are equivalent as per Euler's formula.[13] To convert between rectangular and polar complex numbers, the following conversion formulas are used:

$$a = r\cos\theta$$
$$b = r\sin\theta$$
and therefore $r = \sqrt{a^2 + b^2}$

For the operations of multiplication, division, exponentiation, and finding roots of complex numbers, it is much easier to use polar complex numbers than rectangular complex numbers. In abbreviated form:

- Multiplication: $(r \text{ cis } \theta) * (R \text{ cis } \varphi) = rR \text{ cis } (\theta + \varphi)$
- Division: $\frac{r \text{ cis } \theta}{R \text{ cis } \varphi} = \frac{r}{R} \text{ cis } (\theta - \varphi)$
- Exponentiation (De Moivre's formula): $(r \text{ cis } \theta)^n = r^n \text{ cis } (n\theta)$

Vector calculus

Calculus can be applied to equations expressed in polar coordinates. Let $\mathbf{r}$ be the position vector $(r\cos(\theta), r\sin(\theta))$, with r and θ depending on time t, $\hat{\mathbf{r}}$ be a unit vector in the direction $\mathbf{r}$ and $\hat{\boldsymbol{\theta}}$ be a unit vector at right angles to $\mathbf{r}$. The first and second derivatives of position are

$$\frac{d\mathbf{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}},$$
$$\frac{d^2\mathbf{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

Let $\mathbf{A}$ be the area swept out by a line joining the focus to a point on the curve. In the limit $d\mathbf{A}$ is half the area of the parallelogram formed by $\mathbf{r}$ and $d\mathbf{r}$,

$$dA = \tfrac{1}{2}|\mathbf{r} \times d\mathbf{r}|,$$

and the total area will be the integral of $d\mathbf{A}$ with respect to time.

Applications

Kepler's laws of planetary motion

Further information: Kepler's laws of planetary motion

Polar coordinates provide a natural means of expressing Kepler's laws of planetary motion. Kepler's first law states that the orbit of a planet around a star forms an ellipse with one focus at the center of mass of the system. The equation given above for conic sections may be used to represent this ellipse.

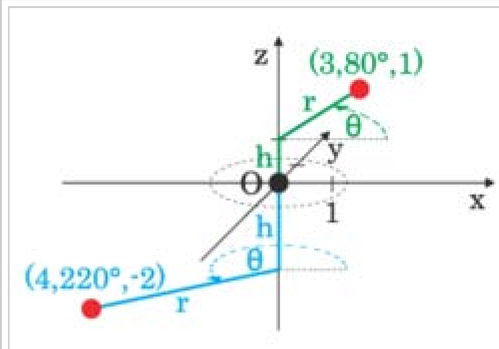
Kepler's second law, the *law of equal areas*, states that *a line joining a planet and its star sweeps out equal are during equal intervals of time*, that is, $\frac{d\mathbf{A}}{dt}$ is constant. These equations can be derived from Newton's laws of motion. A full derivation using polar coordinates is discussed in Kepler's laws of planetary motion.

Three dimensions

The polar coordinate system is extended into three dimensions with two different coordinate systems, the cylindrical and spherical coordinate systems.

Cylindrical coordinates

The *cylindrical coordinate system* is a coordinate system that essentially extends the two-dimensional polar coordinate system by adding a third coordinate measuring the height of a point above the plane, similar to the way in which the Cartesian coordinate system is extended into three dimensions. The third coordinate is usually denoted h, making the three cylindrical coordinates (r, θ, h).



2 points plotted with cylindrical coordinates

The three cylindrical coordinates can be converted to Cartesian coordinates by

$$\begin{aligned} x &= r\,\cos\theta \\ y &= r\,\sin\theta \\ z &= h \end{aligned}$$

Spherical coordinates

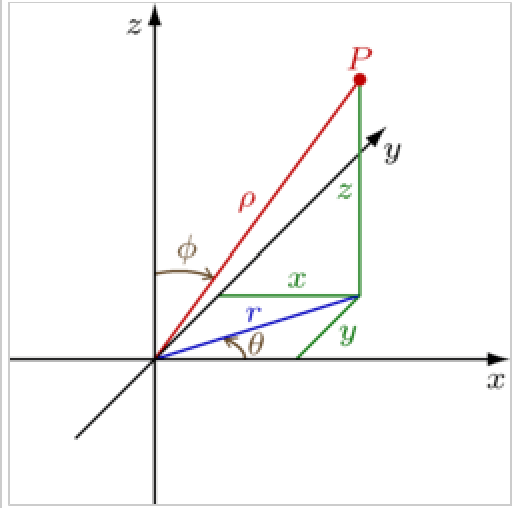
Polar coordinates can also be extended into three dimensions using the coordinates (ρ, φ, θ), where ρ is the distance from the pole, φ is the angle from the z-axis (called the colatitude or zenith and measured from 0 to 180°) and θ is the angle from the x-axis (as in the polar coordinates). This coordinate system, called the *spherical coordinate system*, is similar to the latitude and longitude system used for Earth, with the latitude being the complement of φ, determined by δ = 90° − φ, and the longitude being measured by l = θ − 180°. [14]

The three spherical coordinates are converted to Cartesian coordinates by

$$\begin{aligned} x &= \rho\,\sin\phi\,\cos\theta \\ y &= \rho\,\sin\phi\,\sin\theta \\ z &= \rho\,\cos\phi \end{aligned}$$

See also

- List of canonical coordinate transformations
- Point plotting
- Point (geometry)
- Line (mathematics)
- Plane (mathematics)



A point plotted using spherical coordinates

Other coordinate systems

- Coordinates (mathematics)
- Coordinate systems
- Cylindrical coordinate system
- Curvilinear coordinates
- Orthogonal coordinates
- Elliptic coordinates
- Hyperbolic coordinates
- Stereographic projection
- Parallel coordinates
- Geocentric coordinates

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