

GRAPHING POLAR EQUATIONS

As indicated in the previous unit, polar coordinates can simplify or allow for a more in depth analysis of equations and their graphs. In addition, when an equation is strictly trigonometric in its nature, rectangular coordinates may have little or no meaning to the equation. Polar graphs also provide other benefits to mathematical analysis in terms of following the format and obeying the rules of functions more readily than do their rectangular counter parts. In this unit we will examine the relationships between polar and rectangular graphs and examine a set of special trigonometric equations that are best analyzed by polar coordinates.

The Polar Coordinate System

Identifying and Graphing Polar Lines and Circles

Standard Forms of Parabolas, Ellipses, and Hyperbolas

Using the Graphing Calculator to Graph Polar Equations

Identifying Circles and Parabolas

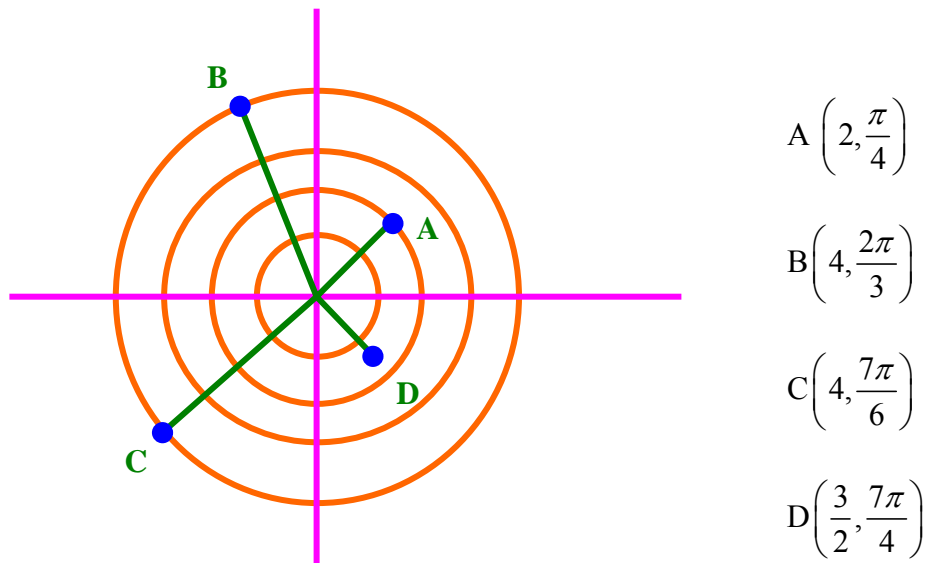
Special Trigonometric Graphs

Table of Signs, Cosines, and Tangents

The Polar Coordinate System

As with rectangular coordinates, polar coordinate points are identified by two components. In rectangular coordinates these components are x and y . In Polar Coordinates the components are called r and θ where r represents a radial distance from the origin and θ , the angle that the radius is “rotated” from the x -axis. Since r represents a radial distance, this value is closely related to the radius of a circle. And since θ can be any angle from 0 to 2π (0 to 360°) – and even more – the polar coordinate system is most commonly represented as a series of concentric circles centered at the origin:

In the diagram below, four polar coordinate points are plotted on this system. (Recall from the last unit that the standard form of a polar point is (r, θ)).



*Note: All angle measures represent positive rotation.

Notice in the above diagram that each concentric circle has a whole number radius and that the location of each point is a point on the circle (or halfway between two circles for point D) that corresponds to the r component of the given point. The second component of the polar point, θ , determines how much angular rotation the radius must undergo before finding its final position on the graph. With the mechanisms of polar coordinates now outlined, we will describe how to graph common rectangular equations on the polar coordinate system.

Identifying and Graphing Polar Lines and Circles

Equations of Lines:

Consider the polar equation: $\theta = \frac{\pi}{4}$. In this example we will seek to determine the rectangular equation of this polar equation to determine the polar graph of the equation.

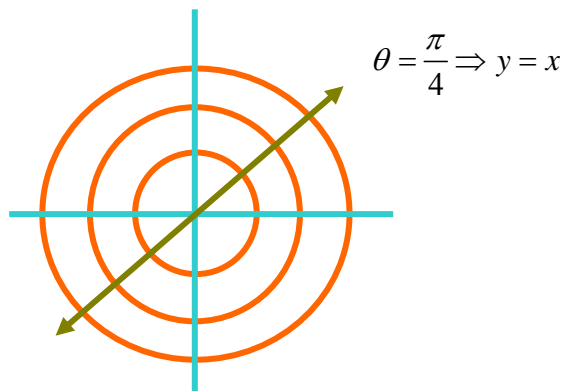
Recall from our definitions of the trigonometric ratios that $\tan \theta = \frac{\text{opp leg}}{\text{adj leg}} = \frac{x}{y}$ for an

inscribed triangle in the unit circle on rectangular coordinates. Using this relationship, we can determine the rectangular form of the above equation in the following manner:

$$\tan \frac{\pi}{4} = 1 \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

$$\Rightarrow \quad \frac{y}{x} = 1 \quad \text{and} \quad y = x$$

The equation $y = x$ is the rectangular equation of a line; therefore, the polar equation $\theta = \frac{\pi}{4}$ is equivalent to the following graph:



For any radial value " r "

Example #1: Determine the rectangular graph of the polar equation:

$$r = \frac{5}{2\sin\theta + \cos\theta}$$

Using techniques developed in the previous unit, we can convert this equation to its rectangular form in the following manner:

Step #1: Cross-multiply and distribute "r"

$$2r\sin\theta + r\cos\theta = 5$$

Step #2: Use the following coordinate identities

$$x = r\cos\theta, \quad y = r\sin\theta \text{ and substitute}$$

$$2y + x = 5$$

This rectangular form of the polar equation is now identified as a line, which can be graphed according to techniques learned in Algebra.

Example #2: Identify the rectangular graph of the polar equation:

$$r = -3\sec\theta$$

Step #1: Recall, $\sec\theta = \frac{1}{\cos\theta}$, which we substitute, then cross-multiply

$$r = -3\left(\frac{1}{\cos\theta}\right)$$

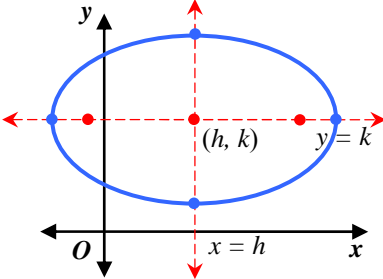
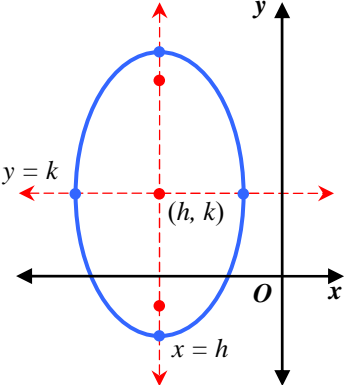
$$r\cos\theta = -3$$

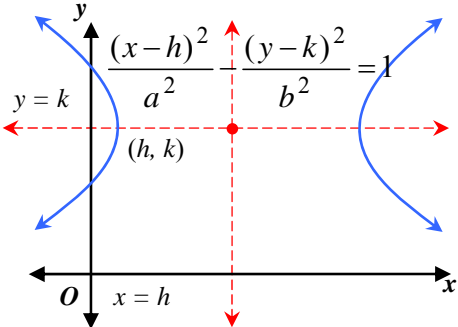
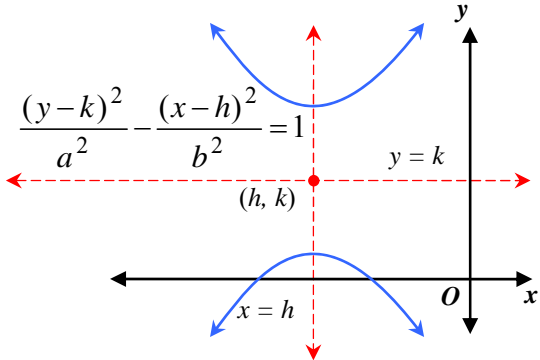
Step #2: Use coordinate identities

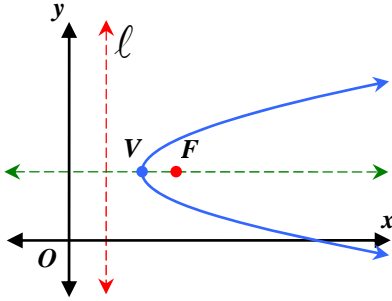
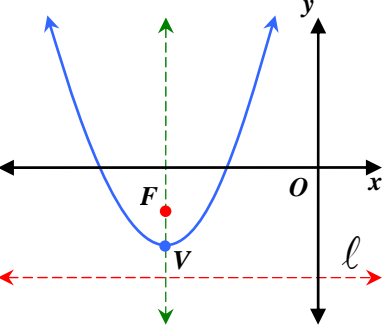
$$x = -3$$

Recalling our rules from Algebra again, the rectangular equation $x = -3$ is the equation of a vertical line located -3 units to the left of the y -axis.

Standard Forms of Parabolas, Ellipses, and Hyperbolas

Standard Form of the Equation of an Ellipse	Orientation	Description
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$ <p>where $c^2 = a^2 - b^2$</p>		<p>Center: (h, k)</p> <p>Foci: $(h \pm c, k)$</p> <p>Major axis: $y = k$</p> <p>Major axis vertices: $(h \pm a, k)$</p> <p>Minor axis: $x = h$</p> <p>Minor axis vertices: $(h, k \pm b)$</p>
$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1,$ <p>where $c^2 = a^2 - b^2$</p>		<p>Center: (h, k)</p> <p>Foci: $(h, k \pm c)$</p> <p>Major axis: $x = h$</p> <p>Major axis vertices: $(h, k \pm a)$</p> <p>Minor axis: $y = k$</p> <p>Minor axis vertices: $(h \pm b, k)$</p>

Standard Form of the Equation of a Hyperbola	Orientation	Description
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$ <p>where $b^2 = c^2 - a^2$</p>		<p>Center: (h, k) Foci: $(h \pm c, k)$ Vertices: $(h \pm a, k)$ Equation of transverse axis: $y = k$ <i>(parallel to x-axis)</i></p>
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1,$ <p>where $b^2 = c^2 - a^2$</p>		<p>Center: (h, k) Foci: $(h, k \pm c)$ Vertices: $(h, k \pm a)$ Equation of transverse axis: $x = h$ <i>(parallel to y-axis)</i></p>

Standard Form of the Equation of a Parabola	Orientation when $p > 0$	Description
$(y - k)^2 = 4p(x - h)$		<p>Vertex: (h, k) Focus: $(h + p, k)$ Axis of Symmetry: $y = k$ Directrix: $x = h - p$ Opening: right if $p > 0$ left if $p < 0$</p>
$(x - h)^2 = 4p(y - k)$		<p>Vertex: (h, k) Focus: $(h, k + p)$ Axis of Symmetry: $x = h$ Directrix: $y = k - p$ Opening: upward if $p > 0$ downward if $p < 0$</p>

Using the Graphing Calculator to Graph Polar Equations

Before continuing to identify the graphs of other polar equations, the last equation $x = -3$ is a vertical line. Technological assistance can be helpful to graph these equations in their polar form. The TI-83+ calculator is set up to graph rectangular equations as functions only. The vertical line equation $x = -3$ is not a function, and therefore cannot be graphed on the calculator in rectangular coordinates. However the TI-83+ is also capable of translating from rectangular to polar coordinates. Perform the following keystrokes to switch the calculator to polar coordinates:

Press **MODE** and then down arrow to the fourth line. On the fourth line right arrow to “Pol” then press **ENTER**. Your calculator should appear as follows:

```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

Now press **Y=** and your graphing editor should now appear as follows:

```
Plot1 Plot2 Plot3
Vr1=
Vr2=
Vr3=
Vr4=
Vr5=
Vr6=
```


The calculator is now set to graph polar equations. (Be sure your calculator is also in “Radian” $\boxed{\text{MODE}}$).

We can now graph the two equations for examples #1 and #2 in the lesson link to “Identifying and Graphing Polar Lines and Circles” as polar equations.

Example #1: Graph the polar equations:

$$r = \frac{5}{2\sin\theta + \cos\theta} : (\text{From Example \#1})$$

and

$$r = -3\sec\theta : (\text{From Example \#2})$$

Step #1: Press $\boxed{Y=}$ and into r_1 type:

$$5/(2\sin\theta + \cos\theta) \text{ and press } \boxed{\text{ENTER}}$$

*Note: “ θ ” can be easily accessed by pressing the $\boxed{\text{X,T},\theta,\text{n}}$ key when in Polar mode.

As we noted in the example, this polar equation translates to the rectangular equation, $2x + y = 5$ and could have been just as easily graphed when the calculator was in “Func” mode.

Step #2: Into r_2 type the polar equation:

$$r = -3\sec\theta$$

*Recall that in order for the calculator to evaluate $\sec\theta$, we need to use the reciprocal of the $\cos\theta$. Type either of the following into r_2 , and then $\boxed{\text{ENTER}}$.

$$r_2 = -3/\cos\theta \quad \text{or} \quad r_2 = -3(\cos(\theta))^{-1}$$

The calculator now graphs the vertical line $x = -3$ as a polar graph. Although the calculator will graph $2x + y = 5$ in “Func” mode, it would not graph $x = -3$ because this is not a function; but, when the equation is written in polar form, the calculator will graph the equation in polar coordinates. As mentioned earlier, this is one advantage to using polar coordinates as the rules of a function are often more readily adapted to polar coordinates. The exact adaptation of those rules is beyond the scope of this course but can be observed as follows.

When the calculator graphed the previous two equations, the student may have noticed that the calculator did not graph each line continuously as it would have in “Func” mode. This is due to the fact that the calculator is evaluating $-3 \cos \theta$ (in the case of r_2) for sequential values of θ beginning at 0 and ending at 2π . If we recall the characteristic curve of, $y = \cos \theta$, we remember that it takes on both positive and negative values alternately throughout its domain. As the calculator plots these values in polar mode, the coordinate of the points on the vertical line alternate between positive and negative values just as $\cos \theta$ alternates in sign for sequential values of θ .

Identifying Circles and Parabolas

In the last unit the Pythagorean Theorem was used to provide the following relationship:

$$r = \sqrt{x^2 + y^2} \Rightarrow r^2 = x^2 + y^2$$

which is the equation of a circle of radius " r " centered at the origin.

Example #1: Identify the polar equation:

$$r = 5$$

Intuitively we recognize that this is the graph of a circle whose radius is 5 and centered at $(0, 0)$. This is seen algebraically by:

$$r = 5$$

$$r^2 = 25$$

$$x^2 + y^2 = 25$$

Example #2: Identify the rectangular graph from the polar equation:

$$r = 4 \sin \theta$$

Step #1: Press and enter

$$r_1 = 4 \sin \theta, \text{ then } \text{GRAPH}$$

From this, we see the polar equation is a circle; however, there is nothing in this equation to indicate the center of the circle. To find the center we change to the rectangular equation.

$$r = 4 \sin \theta$$

$$r^2 = 4(r \sin \theta) : \quad (\text{Multiply both sides by "r"})$$

$$x^2 + y^2 = 4y$$

$$x^2 + y^2 - 4y = 0 \quad (\text{Complete the square on } y)$$

$$x^2 + (y - 2)^2 = 4$$

Therefore $(0, 2)$ is the center and the circle has a radius of $r = 2$.

Example #3: Identify the rectangular graph of:

$$r = 2 \csc \theta \cot \theta$$

Step #1: To use the graphing calculator type in the following

$$r_1 = 2(\sin(\theta))^{-1} * (\tan(\theta))^{-1}$$

We see this is a horizontal parabola that opens to the right.

Step #2: Find the rectangular equation to find the vertex, focus and "a"-value.

$$r = \left(\frac{2}{\sin \theta} \right) \left(\frac{\cos \theta}{\sin \theta} \right) = \frac{2 \cos \theta}{\sin^2 \theta}$$

$$r^2 = \frac{2r \cos \theta}{\sin^2 \theta} \quad (\text{Multiply both sides by "r"})$$

$$r^2 \sin^2 \theta = 2r \cos \theta \quad (\text{Cross multiply})$$

$$(r \sin \theta)^2 = 2r \cos \theta$$

$$y^2 = 2x \quad \Rightarrow \quad x = \frac{1}{2} y^2$$

$$a = \frac{1}{2} \quad V(0, 0) \quad F(2, 0)$$

Special Trigonometric Graphs

As indicated so far, when the calculator is in polar mode, trigonometric equations can be graphed continuously as though they were functions even though they do not pass the vertical line test (VLT). In addition to the equations already described, there are an additional set of trigonometric equations which do not readily translate to rectangular coordinates but can be easily graphed and analyzed using polar coordinates. The following table lists these special curves in their standard polar forms. For the examples in the rest of the unit, and the assignment, we will graph and examine characteristics of these special curves.

Curve:	Standard Equation(s)	
1.) Cardioid	$r = a \pm a \cos \theta$ $r = a \pm a \sin \theta$	$a > 0$
2.) Limacon (Without inner loop)	$r = a \pm b \cos \theta$ $r = a \pm b \sin \theta$	$0 < b < a$
3.) Limacon (With inner loop)	$r = a \pm b \cos \theta$ $r = a \pm b \sin \theta$	$0 < a < b$
4.) Lemniscate	$r^2 = a^2 \cos(2\theta)$ $r^2 = a^2 \sin(2\theta)$	$a > 0$
5.) 3-Petal Rose	$r = a \sin(3\theta)$ $r = a \cos(3\theta)$	$a > 0$
6.) 4-Petal Rose	$r = a \sin(2\theta)$ $r = a \cos(2\theta)$	$a > 0$
7.) Spiral	$r = a$ $a \in (-\infty, 1) \cup (1, \infty)$	$a \neq 0$

*Note: Many forms of these curves may be viewed at <http://www.wikipedia.org/>

Example #1: Identify the following trigonometric graphs: Graph each curve on your calculator on your own to identify each shape. For each example press **WINDOW** and type in any special window setting values. If no window settings are indicated, press **ZOOM**, 6 to view the graph.

Matching:

1.) $r = 5 \sin 2\theta$

A.) Spiral

2.) $r = 3 + 7 \cos \theta$

B.) Lemniscate

3.) $r = \pm \sqrt{25 \cos(2\theta)}$
(Use r_1 and r_2)

C.) 3-Petal Rose

4.) $r = 5 + 4 \sin \theta$
(Set Ymin = -3)

D.) Cardioid

E.) Limacon with inner loop

5.) $r = 5 \sin(3\theta)$

F.) Limacon without inner loop

6.) $5 - 5 \cos \theta$

G.) 4-Petal Rose

Experiment with other variations on these standard equations on your own.

Answers: 1.) G 2.) E 3.) B 4.) F 5.) C 6.) D

Table of Sines, Cosines, and Tangents

Degrees	Radians	cos	sin	tan
0	0	1	0	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	0	1	undefined
120	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
150	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
180	π	-1	0	0
210	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	0	-1	undefined
300	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
330	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
360	2π	1	0	0