Linear Functions

In this lesson you will study linear functions which are the most basic algebraic functions. You will explore relations in which the *y*-value is dependent on the *x*-value for a given set of ordered pairs. You will make tables of *x* and *y* values for linear equations and graph the equation. You will also learn how to find the slope, or steepness, of a line given a diagram. You will analyze how equations change when some values are kept constant while others change. The lesson will conclude with examining some graphs about constant rates of change.

Graphing Linear Equations

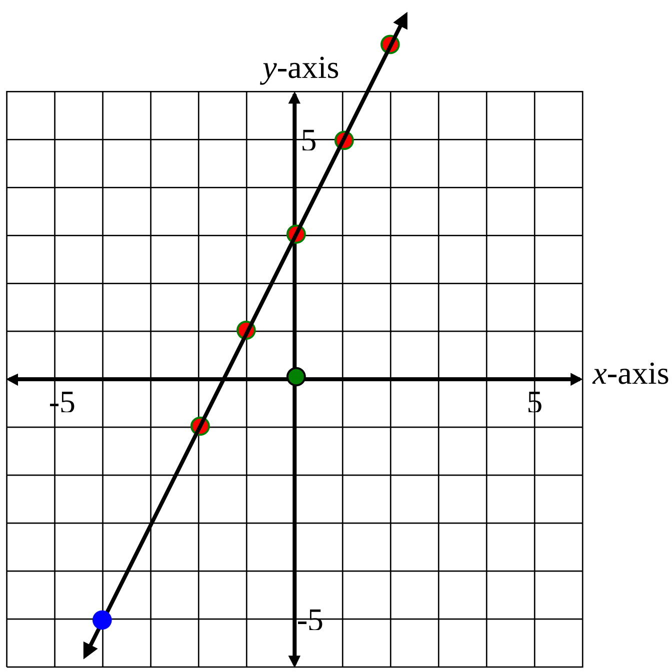
The graph of a linear equation is a straight line.

Graph $y = 2x + 3$

- First organize the data in a table.
- You may choose several values for x. In this example we chose –2 through 2.
- Substitute the values for x in the equation to find y.
- Make ordered pairs with the results.
- Graph the points, draw a straight line through the points.

$y = 2x + 3$			
x	Substitution	y	Ordered Pairs (x, y)
–2	$y = 2 \times (-2) + 3$	-1	(–2, –1)
–1	$y = 2 \times (-1) + 3$	1	(–1, 1)
0	$y = 2 \times 0 + 3$	3	(0, 3)
1	$y = 2 \times 1 + 3$	5	(1, 5)
2	$y = 2 \times 2 + 3$	7	(2, 7)

Use the ordered pairs to graph the function.



Check
Write the ordered pair for a different point on the line by reading the coordinates where the line crosses through the point.

We will use the blue point which is not listed in the table, but falls on the line. Its coordinates are **(-4, -5)**. Substituting –4 for x and –5 for y in the equation gives the following results.

$$\mathbf{y = 2x + 3}$$
$$\mathbf{-5 = 2 \times (-4) + 3}$$
$$\mathbf{-5 = -8 + 3}$$
$$\mathbf{-5 = -5} \quad \textbf{Checks}$$

We have the correct graph for the linear equation, $y = 2x + 3$

Slope

The slope of a line describes the steepness of the line. The slope is the ratio of vertical rise to horizontal run.

(A) To find the slope of a line graphed on a coordinate plane

-identify a point on the line

-from that point move up or down until you are directly across from the next point

-move left or right to the next point

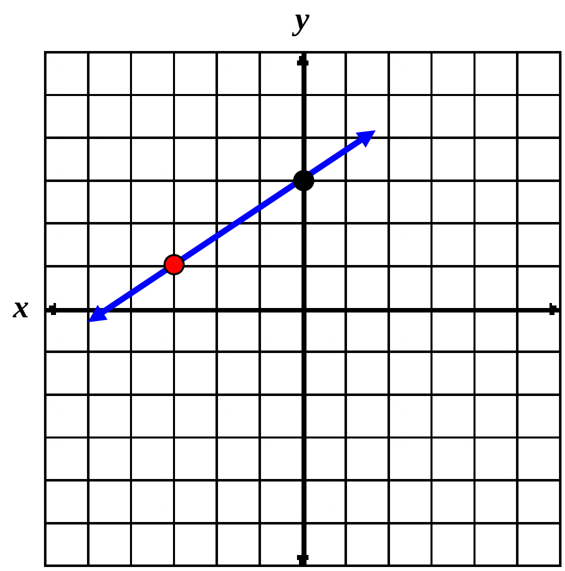
Example: From the graph below determine the slope of the line.

-put your pencil on the red point.

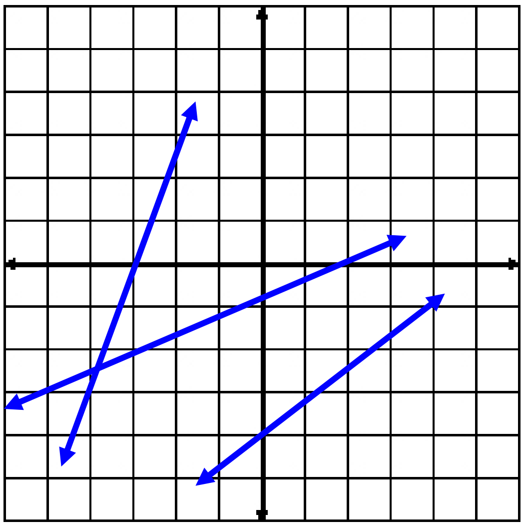
-move straight up (vertical rise) until your pencil is in the same line as the black point, (2 units)

-move right (horizontal run) until you reach the black point. (3 units)

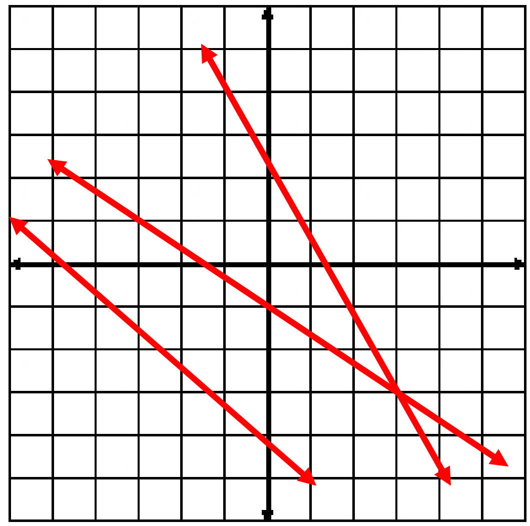
You have now determined the slope of the line to be $\frac{2}{3}$.



On a coordinate plane there are lines that have positive slopes and lines that have negative slopes. Below is an illustration of both.



Lines with positive slopes
rise to the right.



Lines with negative slopes
rise to the left.

At this point we are going to learn how to find the slope of a line by using two points that lie on the line.

B) The definition of slope states that given two points, (x_1, y_1) and (x_2, y_2), the formula for find the slope of a line containing these points is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

*Notice, this is the vertical change over the horizontal change.

Example: Find the slope of the line containing the point A(-2, -6) and B(3, 5).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-6 - 5}{-2 - 3}$$

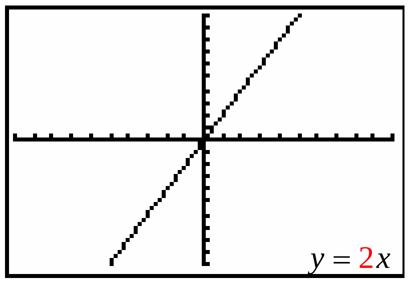
$$m = \frac{-11}{-5}$$

$$m = \frac{11}{5}$$

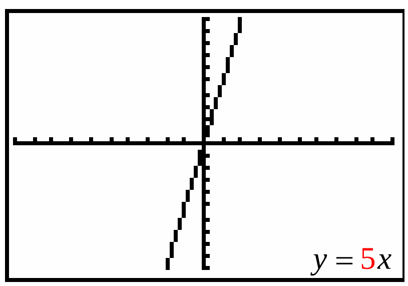
Compare How Changes in an Equation Affect the Related Graph

Once we know what to expect from certain relations that are defined by equations, we can experiment with changing the equations and predicting how those changes will appear "graphically".

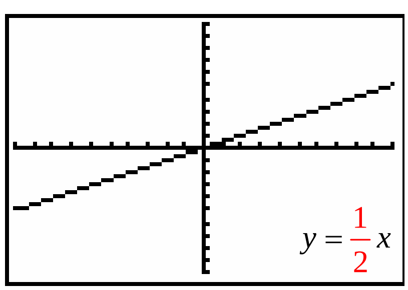
Let's begin with the graph, $y = 2x$.



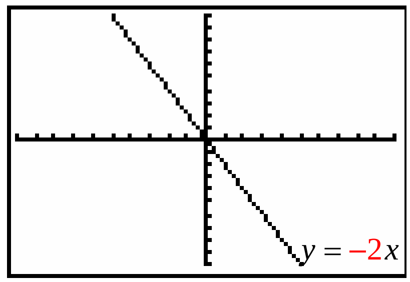
Now, let's increase the coefficient from 2 to 5, $y = 5x$. The line is steeper.



Next, let's decrease the coefficient from 5 to 0.5, $y = \frac{1}{2}x$. The line is flatter (less steep).



Finally, let's change the coefficient from 0.5 to –2, $y = -2x$. The line changes direction, up and to the left.

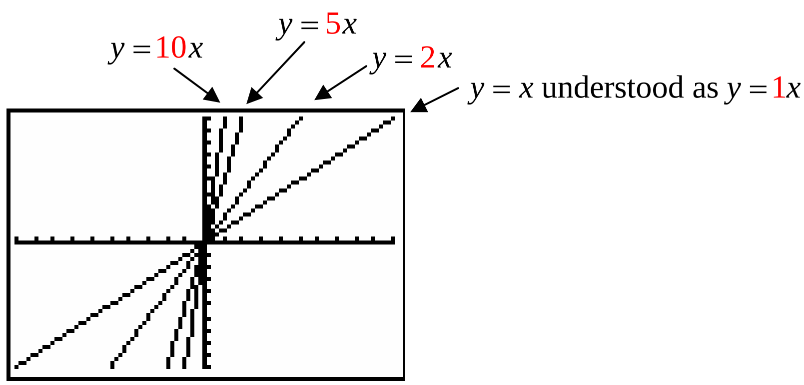


This demonstrates the effect on a line called "slope". We can see larger values greater than one will make steep lines; small fractional values make flatter lines; and negative values will point the line in the opposite direction. All of these changes occur as we change the coefficient of the "*x*".

Now let's compare four equations as the coefficient of "*x*" increases.

Graph the given equations:

$y = x$
$y = 2x$
$y = 5x$
$y = 10x$



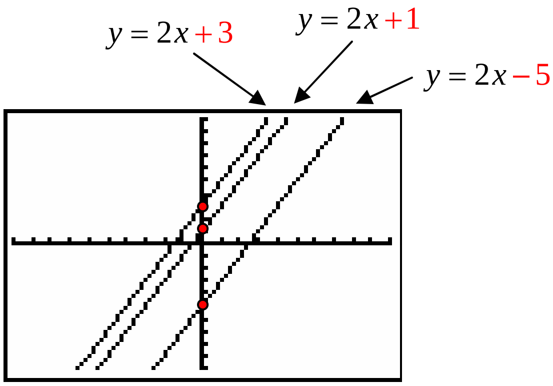
This graph shows that the lines appear to grow steeper as the coefficient increases.

Now graph this set of given equations:

$y = 2x + 1$

$y = 2x + 3$

$y = 2x - 5$



Notice all of the equations have the same slope of two, but that the line slides up and down the *y*-axis. The graphs of the equations show that adjusting the constant value will translate the line up or down the *y*-axis.

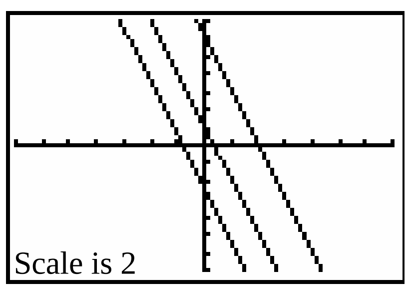
Now practice observing the changes in the graphs. If you have a graphing calculator available, it would be best to enter the equations and watch as they are graphed. The first equation will be graphed first, and so on. If you do not have a graphing calculator, then create tables of values and make the graphs on grid paper.

> *Problem 1*: What remains the same in all three equations? What changes occur in the graphs of the three equations? (Note: The scale on the graphing calculator is set to two instead of one.)

$y = -3x + 2$

$y = -3x - 5$

$y = -3x + 12$

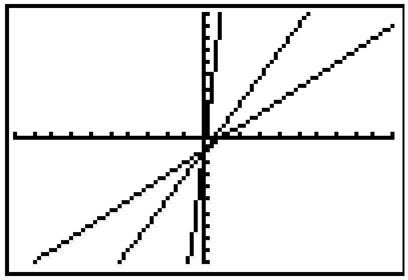


Problem 2: What remains the same in all three equations? What changes occur in the graphs of the three equations?

$y = x - 1$

$y = 2x - 1$

$y = 12x - 1$



Answers

Problem 1: The slope is the same, –3. The graph moves up and down the *x*-axis depending on the constant value. $y = -3x + 12$ crosses the *x*-axis at (0,12), $y = -3x + 2$ crosses the *x*-axis at (0, 2), and $y = -3x - 5$ crosses the *x*-axis at (0, –5).

Problem 2: The *y*-intercept is the same for all three equations, (0, –1). As the coefficient of *x* (slope) increases, so does the steepness of the line. $y = x - 1$ is the less steep of the three equations, while $y = 12x - 1$ is the most steep.

Note: All of the equations used in this lesson were written in the slope-intercept form, $y = mx + b$, where "*m*" represents the slope and "*b*" represents the *y*-intercept.

Slope-Intercept Form

One way of graphing the equation of a line is by using the slope-intercept form which identifies the slope and the *y*-intercept.

Slope-Intercept Form

$$y = mx + b$$

Where *m* represents the slope and *b* represents the *y*-intercept, the point at which the graph crosses the *y*-axis.

Example 1: Identify the slope and *y*-intercept.

$$y = \frac{-2}{3}x - 4$$

$$\text{slope} = \frac{-2}{3}$$

$$y\text{-intercept} = -4 \qquad \textit{or} \qquad (0, -4)$$

To graph a line using the slope and *y*-intercept

1) Arrange the equation into the form $y = mx + b$. (This means solve any equation for *y*.
2) Identify the *y*-intercept and plot the point (0, *b*).
3) Use the $\frac{\text{rise}}{\text{run}}$ ratio for slope to plot more points.
4) Draw a line through the points with a straight edge.

Example 2: Graph $-3x + 2y = -6$

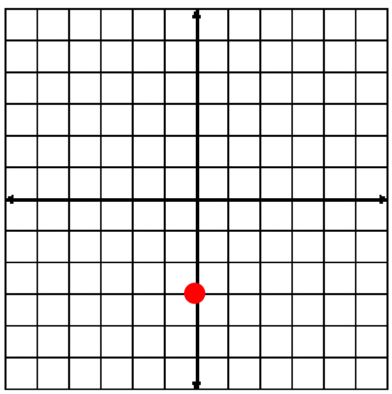
1) Solve the equation for y.

$-3x + 2y = -6$ add $3x$ to both sides

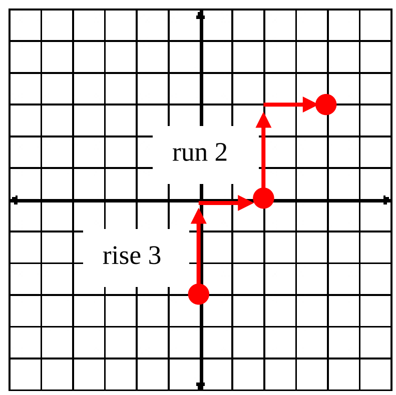
$2y = 3x - 6$ divide all terms by 2

$$y = \frac{3}{2}x - 3$$

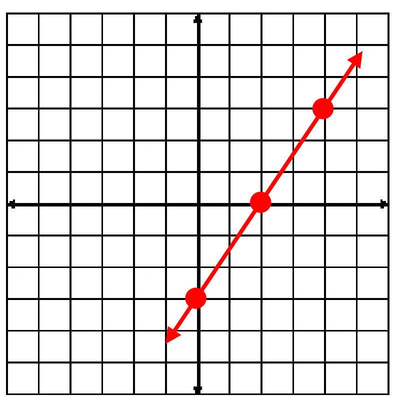
2) Plot the y-intercept, (0, –3).



3) Use the ratio of $\frac{\text{rise}}{\text{run}}$ and the slope $\frac{3}{2}$ to plot more. points



4) Draw a line through the points with a straight edge.



Graphing a Line Using the x- and y-intercepts

Another way to graph linear equations is by using the x- and y-intercepts. In the last section you learned that the y-intercept is the point at which a line crosses the y-axis. The x-intercept is the point at which the line crosses the x-axis.

To find the x- and y-intercepts:

1) Replace x with 0 in the equation and solve for y to locate the y-intercept $(0, y)$.

2) Replace y with 0 in the equation and solve for x to locate the x-intercept $(x, 0)$.

3) Plot the two points and connect them with a straight edge.

Example: Graph $2x – 3y = 6$ by using the x- and y-intercepts.

1) $2(0) – 3y = 6$

$–3y = 6$

$y = –2$

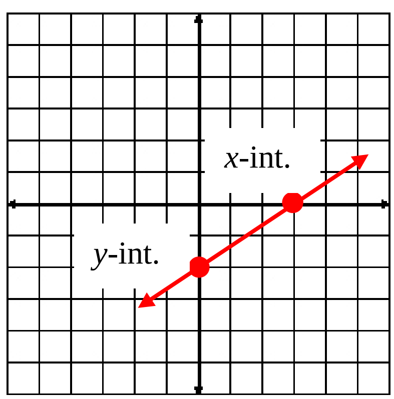
y-intercept = $(0, –2)$

2) $2x – 3(0) = 6$

$2x = 6$

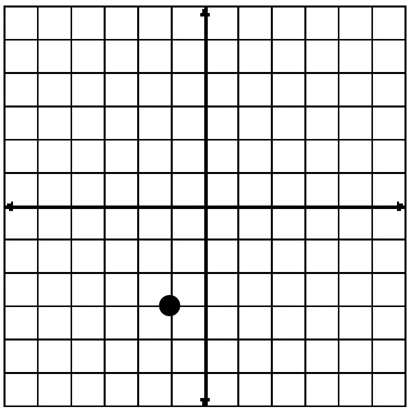
$x = 3$

x-intercept = $(3, 0)$

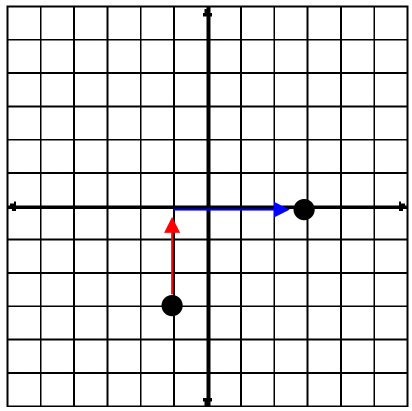


Graphing a Line on a Coordinate Plane Using a Point and the Slope

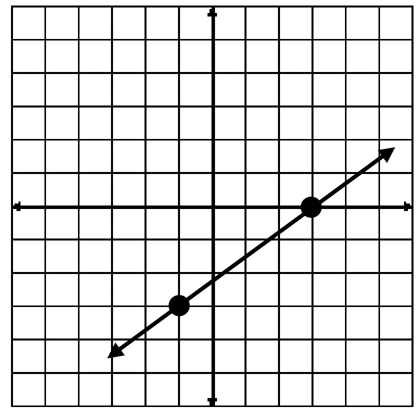
Example: Graph the line containing the point (–1, –3) and having a slope of $m = \dfrac{3}{4}$.



1. Plot the point (-1, -3)



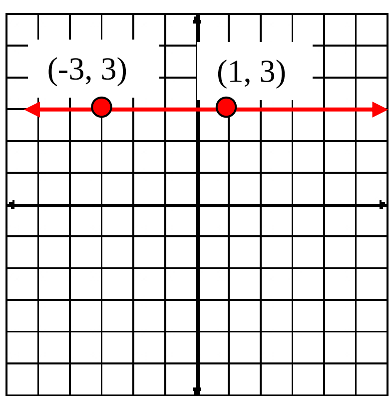
2. Use the rise (3 units) over run (4 units) ratio for slope to plot a second point.



3. Draw a line through the points with a straightedge.

There are two more types of lines that we need to discuss. They are the vertical and horizontal lines that have special types of slopes.

Example 1:



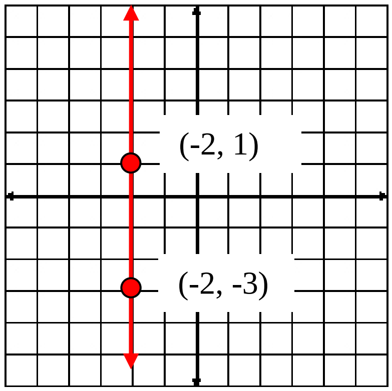
The *y*-coordinate for every point on a horizontal line is the same.

Choose two points on the line and use the slope formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to determine the slope.

$$m = \frac{3-3}{-3-1} = \frac{0}{-4} = 0$$

The slope of every horizontal line is 0.

Example 2:



The *x*-coordinate for every point on a vertical line is the same.

Choose two points on the line and use the slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, to determine the slope.

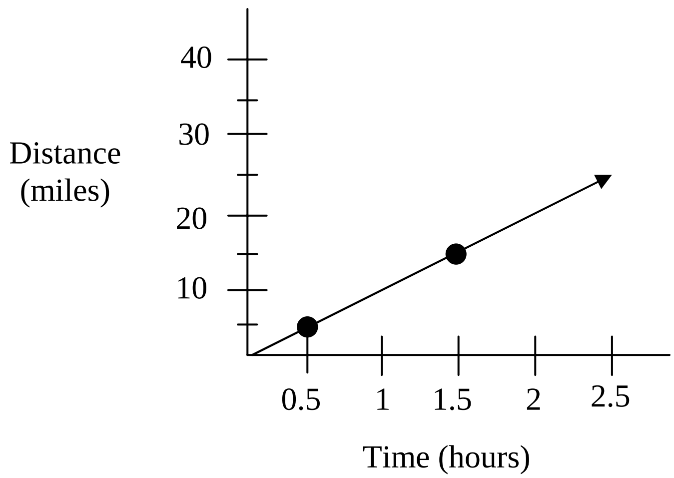
$$m = \frac{1-(-3)}{-2-(-2)} = \frac{4}{0} = \text{ undefined or no slope.}$$

Rate of Change

Rate of change is directly related to slope and can be found using the following:

$$\text{rate of change} = \frac{\text{change in distance}}{\text{change in time}}$$

Example: The graph shows the distance a bicyclist travels at a constant speed. Find the speed of the bicyclist.



Select two points and use the slope formula.

Let (x_1, y_1) = (0.5, 5)

Let (x_2, y_2) = (1.5, 15)

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{15-5}{1.5-0.5} = \frac{10 \text{ miles}}{1 \text{ hour}}$$

The bicyclist travels 10 miles for every hour he travels.

Constant Rates of Change and Predicting Solutions

In the problem below, we will examine a constant rate of change and use that rate to make a prediction.

Example 1: On a trip across country, the Wilson family was able to travel an average of 50 miles per hour for several days. They drove nine hours per day and spent the other 15 hours per day sightseeing and sleeping. Establish a table to determine how many miles, days, and hours were spent on the two week trip.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hours driving	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
Miles ($D = 50h$)	450	900	1350	1800	2250	2700	3150	3600	4050	4500	4950	5400	5850	6300	6750

Based on data in the table, we can write relationships and equations about the data.

To find the hours driving, we can multiply the number of days times 9.

$$H = 9d$$

To find the miles traveled, we can multiply 9 × number of days × 50.

$$M = 9(50)d$$

Using this chart and a little algebra, we can predict much about a long trip or a similar situation like this one.

Example 2: Answer the follow questions about the table and relationships discussed in the previous example.

Predict how many hours the Wilson family would drive after 20 days?

$$H = 9d$$
$$H = 9(20)$$
$$H = 180$$

The Wilson family would drive 180 hours in 20 days.

Predict how many miles the Wilson family would travel in 20 days.

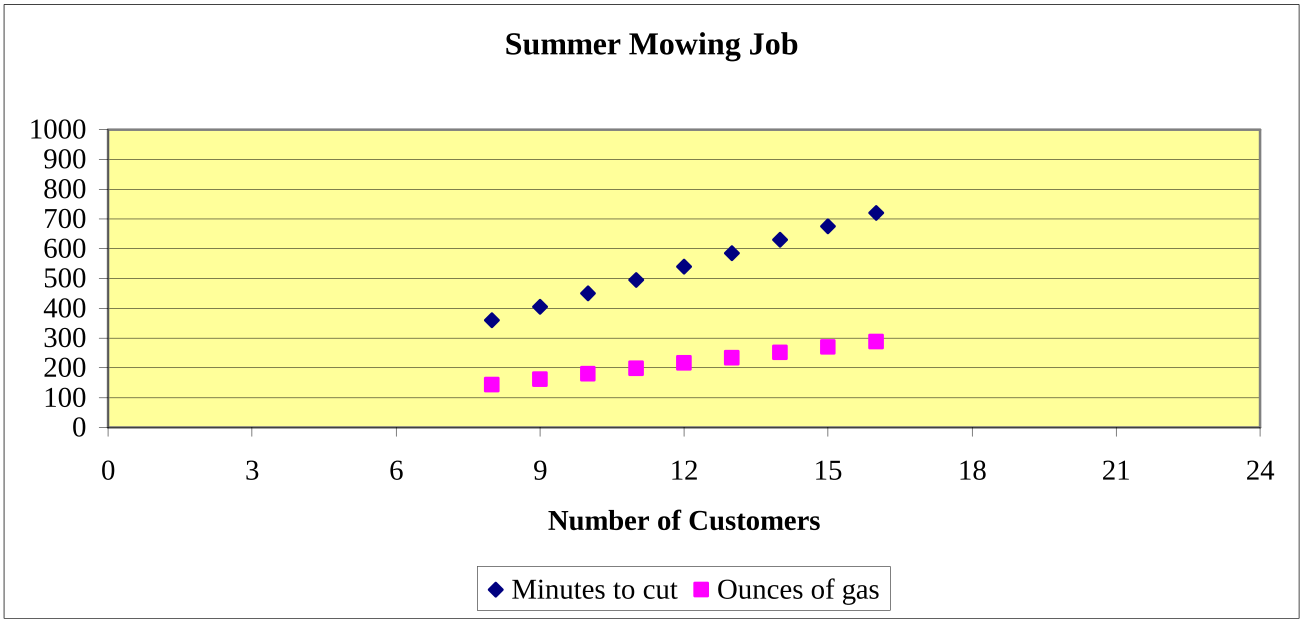
$$M = 9\left(50\right)d$$
$$M = 9(50)(20)$$
$$M = 9000$$

The Wilson family would travel 9000 miles in 20 days.

Example 3: Set up a chart to analyze Joe's work on his summer job. Joe mows lawns for customers in his neighborhood. He will spend 45 minutes mowing each lawn and use about 18 ounces of gasoline per lawn. Joe wants to estimate his work schedule and load for 8 through 16 customers.

Customers	8	9	10	11	12	13	14	15	16
Minutes to cut	360	405	450	495	540	585	630	675	720
Ounces of gas	144	162	180	198	216	234	252	270	288

Joe can now use the chart and graph to decide how many customers to service this summer.



Notice how the values increase as the number of customers increase. This prediction is a positive correlation as well as a positive slope.

If Joe had 22 customers, how many minutes would he spend mowing grass? How much gasoline would he need?

Since he is allowing 45 minutes per customer, you could write the following relationship: Number of minutes = 45 × number of customers.

$M = 45\ C$
$M = 45\ (22)$
$M = 990$ minutes

Joe would spend 990 minutes mowing for 22 customers.

Since he is allotting 18 ounces of gasoline per lawn, he could write the following relationship: number of gallons of gasoline = 18 × number of customers.

$G = 18 \times C$
$G = 18 \times 22$
$G = 396$

Joe would use 396 ounces of gasoline for 22 customers.

In the problem set, you will be asked to make predictions based on the analysis of the data given.