POLAR COORDINATES, EQUATIONS, AND GRAPHS

In a previous unit we introduced the concept that the location of a complex number can be found by rotating that number's vector through an angle on the complex plane. In a later unit, the trigonometric coordinate identities were introduced which state the following:

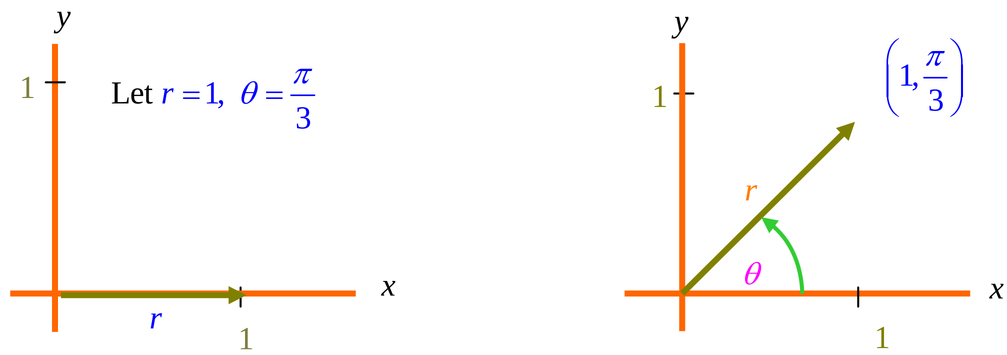
> If (x, y) is a point on the xy-plane, then $x = r\cos\theta,\ y = r\sin\theta$,
> and $(r\cos\theta, r\sin\theta)$ is the trigonometric representation of that point.

In this unit we combine these two concepts to introduce the graphing technique called "Polar Coordinates" and demonstrate how this graphing system allows us to investigate many different functions which could not be as easily analyzed in the traditional "rectangular coordinate plane".

The Polar Coordinates of a Point

To locate a point on the xy-plane, two components of that point are needed to fix that points' location. These two components are the x and y values. To find any point on the xy-plane (except axis points), one begins at the origin, and then moves either left or right according to the value of the x-component, then make a right angle, and proceed vertically according to the y-component of the point. This straight line, right angle motion provides the name "rectangular coordinates" for any point on the plane. These components will allow us to easily accommodate the geometry of complex numbers and trigonometry into the investigations.

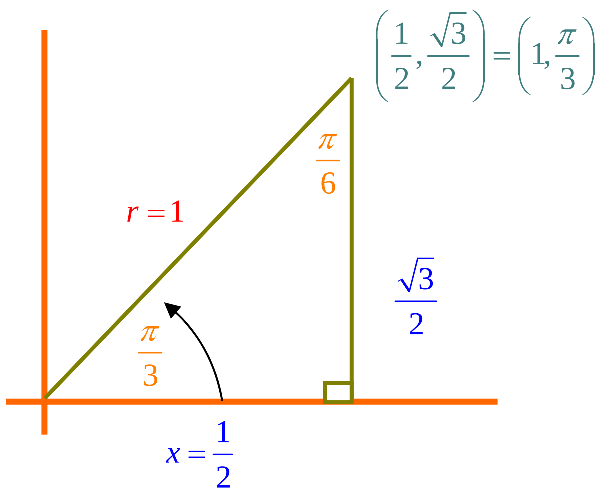
The standard form of a Polar Coordinate is given by (r,θ), where "r" represents the radius of a circle, centered at the origin and "θ" represents an angle measure used to rotate "r" to a location on the graph. In the diagram below, the method for locating a point on the graph by polar coordinates is demonstrated.



Notice that in the first diagram, a choice for r is made, and then measured on the x-axis. Next, r is rotated through the angle, $\frac{\pi}{3}$. When the endpoint of "r" reaches its final location, it is labeled according to the values of 'r' and θ, which determined its location on the xy-plane $\left(1,\frac{\pi}{3}\right)$.

Notice that a polar point is still located on the xy-plane as are all rectangular representations of points. In fact any point that can be represented through rectangular coordinates can also be converted to a polar representation and vice versa. In the above diagram, you may have already noticed that by choosing $r=1$ and $\theta=\frac{\pi}{3}$, we have selected the radius of the unit circle and the critical angle $\frac{\pi}{3}=60°$ from a 30-60-90 inscribed triangle discussed in a previous unit.

The diagram below overlays a 30-60-90 triangle on the previous diagram with the side lengths indicated and rectangular coordinates of the same point also labeled.



From this last diagram, we see that the polar coordinate: $\left(1, \frac{\pi}{3}\right)$ is equivalent to the rectangular coordinate: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Just as $\frac{1}{3}$ and $\frac{19}{57}$ are equivalent fractions, the coordinates $\left(1, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ are equivalent representations of the same point on the xy-plane. Over the next two units, the importance of representing a rectangular coordinate as a polar coordinate should become apparent. We begin first by learning the rules for converting and graphing rectangular to polar coordinates and vice versa.

Converting Between Polar and Rectangular Coordinates

Recall that coordinate identities for trigonometry are:

$$x = r\cos\theta \quad : \quad y = r\sin\theta$$

By using these identities, we can convert freely between polar and rectangular coordinates in the following manner:

Example #1: Convert the following polar coordinates to rectangular coordinates.

a.) $\left(2, \frac{\pi}{4}\right)$ b.) $\left(3, \frac{5\pi}{6}\right)$ c.) $\left(\frac{1}{2}, \pi\right)$

For each point, the following components are listed:

$$r = 2 \Rightarrow \theta = \frac{\pi}{4}$$

$$r = 3 \Rightarrow \theta = \frac{5\pi}{6}$$

$$r = \frac{1}{2} \Rightarrow \theta = \pi$$

By substituting these values into the coordinate identities, we obtain the following rectangular coordinates:

a.) $x = 2\cos\frac{\pi}{4} : y = 2\sin\frac{\pi}{4} \Rightarrow \left(2\cdot\frac{\sqrt{2}}{2}, 2\cdot\frac{\sqrt{2}}{2}\right) = (\sqrt{2}, \sqrt{2})$

b.) $x = 3\cos\frac{5\pi}{6} : y = 3\sin\frac{5\pi}{6} \Rightarrow \left(3\cdot\left(-\frac{\sqrt{3}}{2}\right), 3\cdot\frac{1}{2}\right) = \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

c.) $x = \frac{1}{2}\cos\pi : y = \frac{1}{2}\sin\pi \Rightarrow \left(\frac{1}{2}(-1), \frac{1}{2}\cdot 0\right) = \left(-\frac{1}{2}, 0\right)$

Example #2: Convert the following rectangular coordinates to polar coordinates

$Part\ A.)\ (2,\ -2)$ $\qquad$ $Part\ B.)\ (-1, -\sqrt{3})$

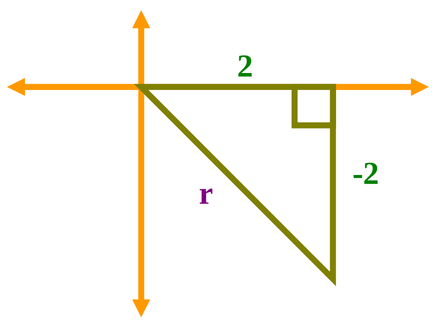
For each point, the following components are listed.

$Part\ A)\ x = 2 : y = -2$ $\qquad$ $Part\ B)\ x = -1 : y = -\sqrt{3}$

By substitution we obtain the following:

$Part\ A.)\ 2 = r\cos\theta : -2 = r\sin\theta$ $\qquad$ $Part\ B.)\ -1 = r\cos\theta : -\sqrt{3} = r\sin\theta$

Each of the above equations contains two variables; "r" and "θ". In order to eliminate one of these variables, we recall that a rectangular point is found by tracing a right triangle on the xy-plane. The diagram below defines the triangle traced by the point in *Part A.*) of the example problem.



Part A: To find the value of *r*, we use the Pythagorean Theorem:

$$r^2 = 2^2 + (-2)^2 = 4 + 4 = 8$$

$$\Rightarrow r = 2\sqrt{2}$$

Part B: To find the value of *r*:

$$r^2 = (-1)^2 + (-\sqrt{3})^2 = 1 + 3 = 4$$

$$\Rightarrow r = 2$$

Once r is found, we can substitute this value into each equation, and then find θ by using the arcsine, arccosine or arctangent.

Part A

$$2 = 2\sqrt{2}\cos\theta$$

$$\frac{\sqrt{2}}{2} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\theta = \frac{\pi}{4}, \frac{7\pi}{4}$$

Part B

B.) $-1 = 2\cos\theta$

$$-\frac{1}{2} = \cos\theta$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

and

Part A

$$-2 = 2\sqrt{2}\sin\theta$$

$$-\frac{\sqrt{2}}{2} = \sin\theta$$

$$\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

$$\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$$

Part B

$$-\sqrt{3} = 2\sin\theta$$

$$\frac{-\sqrt{3}}{2} = \sin\theta$$

$$\theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$$

To determine the final polar coordinates of each point, we refer back to our original rectangular coordinates.

In Part A, the point $(2, \ -2)$ is a "Quadrant IV" point. For the angles found using the arccosine and arcsine above, only $\theta = \frac{7\pi}{4}$ is found in quadrant IV of the graph. Therefore the polar coordinates for the rectangular point $(2, \ -2)$ is $(r, \theta) = \left(2\sqrt{2}, \frac{7\pi}{4}\right)$.

In Part B, $(-1,-\sqrt{3})$ is a Quadrant III point. For the angles found, only $\frac{4\pi}{3}$ is found in Quadrant III of the graph. Therefore for the rectangular point $(-1,-\sqrt{3})$, the polar coordinate is $(r,\theta) = \left(2,\frac{4\pi}{3}\right)$.

Transforming Polar and Rectangular Equations

Recall that the following is an equation of a circle:

$$x^2 + y^2 = 25$$

This equation is given in rectangular coordinates. Using our trigonometric coordinate identities, this equation can be transformed into an equivalent polar equation for a circle in the following manner:

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 25$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 25$$

$$r^2(\cos^2\theta + \sin^2\theta) = 25$$

$$r^2 \cdot 1 = 25$$

$$r^2 = 25$$

$$r = 5$$

Notice that by using polar coordinates, the rectangular equation for this circle is greatly simplified. In fact, the polar equation $r = 5$ fits the geometric definition of a circle more closely than does the rectangular equation ($r = 5 \Rightarrow$ a circle whose radius is 5). Depending on the application, polar equations are often simpler to manipulate or more descriptive of the mathematical situation. Still both equation formats have important uses and choosing the best form of an equation depends on the application involved, the information needed (or given), and even the ability of the student. For the rest of this unit and in the assignment, practice problems and techniques will be used to transform equations from polar to rectangular and rectangular to polar.

Example #1: Transform the equation $4xy = 9$ to Polar form

Step #1: Substitute the coordinate identities for x and y, then simplify.

$$4(r\cos\theta)(r\sin\theta) = 9$$

$$4r^2\cos\theta\sin\theta = 9$$

Recall that $2\sin\theta\cos\theta = \sin 2\theta$ is a double angle identity.

Step #2: $2r^2(2\cos\theta\sin\theta) = 9$

$$2r^2 \sin 2\theta = 9$$

Example #2: Transform the following polar equation to rectangular form

$$r = 6\cos\theta$$

Since r represents the radius of a circle, we know that it must be a positive real number $(r \in \mathbb{R}^+)$. For this reason the equation must obey the laws and properties of Real numbers including the multiplication property of equality. Using this property, we can transform the equation in the following manner.

$$r = 6\cos\theta$$

$$r \cdot r = 6 \cdot r\cos\theta$$

$$r^2 = 6x$$

We also know that $x^2 + y^2 = r^2$ according to the Pythagorean Theorem. Therefore:

$$r^2 = 6x$$

$$x^2 + y^2 = 6x$$

$$x^2 - 6x + y^2 = 0$$

Next we complete the square on x

$$x^2 - 6x + 9 + y^2 = 9$$

$(x-3)^2 + y^2 = 9$ This is the rectangular equation of a circle.

Example #3: Transform $x^2 = 4y$ to polar form

$$(r\cos\theta)^2 = 4r\sin\theta$$

$$r^2\cos^2\theta = 4r\sin\theta$$

$$r^2\cos^2 - 4r\sin\theta = 0$$

Although we may be tempted to use the identity: $\cos^2\theta = 1-\sin^2\theta$ as a substitution, this only results in transforming the two-term expression in to a three-term expression:

$$r^2(1-\sin^2\theta)-4r\sin\theta = 0$$

$$r^2 - r^2\sin^2 - 4r\sin\theta = 0$$

Even though this now appears that a further identity may be used, all such attempts do not significantly simplify the problem. We are left to conclude that the most reasonable transformation for $x^2 = 4y$ is

$$r^2\cos^2\theta - 4r\sin\theta = 0$$

Example #4: Transform the following equation to rectangular form:

$$r = \frac{4}{1-\cos\theta} \qquad \text{cross multiply}$$

$$r - r\cos\theta = 4$$

$$r - x = 4$$

$$r = x+4 \qquad \text{Now: } r = \sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = x+4$$

$$x^2+y^2 = (x+4)^2$$

$$x^2+y^2 = x^2+8x+16$$

$$y^2 = 8x+16 \qquad \text{(Horizontal parabola)}$$

Example #5: Transform the following to rectangular form:

$$r^2 \sin 2\theta = 8$$

$$r^2 (2 \sin\theta \cos\theta) = 8$$

$$r^2 (\sin\theta \cos\theta) = 4$$

$$r \sin\theta r \cos\theta = 4$$

$$xy = 4$$

Example #6: Transform the following to polar form:

$$(x-2)^2 + (y+3)^2 = 25$$

$$(r\cos\theta - 2)^2 + (r\sin\theta + 3)^2 = 25: \quad \text{FOIL each quantity}$$

$$r^2\cos^2\theta - 4r\cos\theta + 4 + r^2\sin^2\theta + 6r\sin\theta + 9 = 25$$

$$r^2\sin^2\theta + r^2\sin^2\theta + 6r\sin\theta - 4r\cos\theta = 12$$

$$r^2(\sin^2\theta + \cos^2\theta) + 6r\sin\theta - 4r\cos\theta = 12$$

$$r^2 \cdot 1 + 6r\sin\theta - 4r\cos\theta = 12$$

$$r^2 - 12 + 6r\sin\theta - 4r\cos\theta = 0$$

Table of Sines, Cosines, and Tangents

Degrees	Radians	cos	sin	tan
0	0	1	0	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	0	1	undefined
120	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
150	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
180	π	-1	0	0
210	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	0	-1	undefined
300	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
330	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
360	2π	1	0	0

Polar and Rectangular Coordinates