# THE AREA OF TRIANGLE AND SOLVING TRIGONOMETRIC EQUATIONS

In the first part of this unit we will explore the use of trigonometry to find the area of a triangle. In the second part of the unit we return to using algebraic techniques in order to find angular solutions to trigonometric equations which are represented in algebraic form. Key to these solutions will be identifying the “domain” of each equation prior to seeking any solution that may restrict possible answers according to how trigonometric functions are defined.

## The Trigonometric Area of a Triangle

The well-known formula for finding the area of a triangle is

$$A = \frac{1}{2}bh$$

Where:

***"A"*** is the area, given in square units

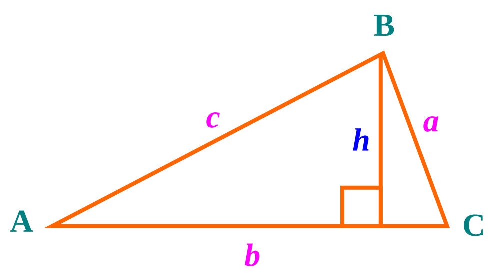
***"b"*** is the base length of the triangle

***"h"*** is an altitude drawn perpendicular to the base

In your previous studies of Geometry, the use of the triangle area formula probably utilized only diagrams and situations that involved side and altitude lengths without incorporating the relationships between sides and the angle measures as is found in trigonometry. Trigonometric area formulas can be very useful in situations where it is impractical to measure the lengths for all sides of a triangle – as in lengths that may be miles long. However, because trigonometry establishes a **"unique"** relationship between an angle's measure and the ratio of the sides of a triangle – regardless of the lengths involved – very large or otherwise inaccessible side lengths can be found with a minimum of factual information. Large triangular areas can become known. In order to fully understand the usefulness of these trigonometric area formulas, we will first begin by demonstrating how the formulas are derived in Geometry.

### Trigonometric Area of Triangle

Consider the following diagram:




According to the traditional formula

$A = \frac{1}{2}bh$ gives the area of the triangle.

However if the value of "***h***" is unknown, but the values of **"*c*"** and **"*b*"** are known, then, the area of the triangle can still be found when the measure of $\angle A$ is also established.

In the above diagram, $A = \frac{1}{2}bh$ but also:

$$\sin \angle A = \frac{h}{c} \;\Rightarrow\; h = c \sin \angle A$$

by substitution $A = \frac{1}{2}bh \;=\; \frac{1}{2}b(c \sin \angle A)$

$$A = \frac{1}{2}bc \sin \angle A$$

By using a similar argument for $\angle B$, $\angle C$, and side **"*a*"** we can establish the following three formulas that can be used to find the area of any triangle through trigonometric means:

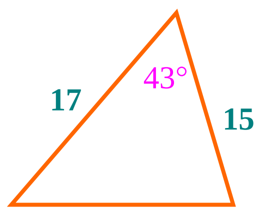
$$A = \frac{1}{2}bc \sin \angle A \qquad A = \frac{1}{2}ac \sin \angle B \qquad A = \frac{1}{2}ab \sin \angle C$$

The key to each of these formulas is that the angle measure "included" between the two known sides is used.

*Example #1*: Find the area of the following triangle from the information given

By any of the above formulas we have:

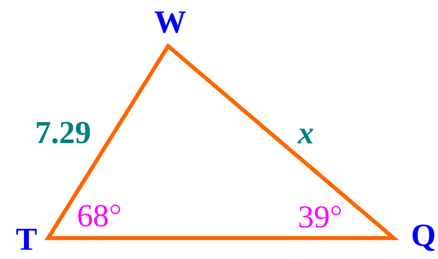
$A = \frac{1}{2}(17)(15)\sin 43 = 86.955$ square units



*Example #2*:

The formula to find the area of this triangle is:

$$A = \frac{1}{2}(7.29) \cdot x \cdot \sin \angle W$$




Since this equation involves three unknowns – **"A"**, **"x"**, and $"m\angle W"$ - we need to discern the values of **"x"** and $"m\angle W"$ in order to proceed.

*Step #1*: Find $m\angle W$. Recall that the angle sum of any triangle = 180°.

Therefore: $m\angle W$ = $180 - 39 - 68 = 73°$

*Step #2*: Use the "Law of Sines" to find the length of ***x***.

$$\frac{x}{\sin 68} = \frac{7.29}{\sin 39} \Rightarrow x = \frac{7.29(\sin 68)}{\sin 39} = 10.74$$

*Step #3*: Find Area of $\triangle WTQ$

$$A = \frac{1}{2}(7.29)(10.74)(\sin 73°) = 37.44 \text{ square units}$$

## Solving Trigonometric Equations

Over the past units we have frequently dealt with trigonometric expressions algebraically. Although not explicitly mentioned before, an expression of the form $\sin\theta$, $\cos\theta$, etc. is a *variable quantity* in the variable "$\theta$" and can therefore be manipulated according to the common laws and properties of Algebra. However, because a trigonometric variable represents a unique geometric relationship between right triangles and the unit circle, solutions to algebraic equations involving trigonometric quantities are subject to special interpretation in order for the solution candidates to make sense. The most common interpretation for these problems involves the domain of trigonometric functions associated with an expression.

Consider the following example:

*Example #1*:

Solve for $\theta$. $\quad 2\sin\theta + 1 = 0$

We proceed by letting $\sin\theta = x$.

$$2x + 1 = 0 \Rightarrow x = -\frac{1}{2} \text{ or } \sin\theta = -\frac{1}{2}$$

From a previous unit, we know that $\sin\theta = -\frac{1}{2}$ when $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\quad (210°, 330°)$ However we have also noted that all trigonometric functions are periodic from $(-\infty, \infty)$. Therefore, for $\sin\theta = -\frac{1}{2}$, $\theta$ can take on an infinite number of values. For example:

$$\theta = 690°\left(\frac{23\pi}{6}\right) \Rightarrow \sin\theta = -\frac{1}{2}$$

$$\theta = -30°\left(-\frac{\pi}{6}\right) \Rightarrow \sin\theta = -\frac{1}{2}$$

$$\theta = 2370°\left(\frac{79\pi}{6}\right) \Rightarrow \sin\theta = -\frac{1}{2}$$

$$\theta = 210° + 360°k \;\; where \;\; k \in \mathbb{N}$$

$$\left(\frac{7\pi}{6} \pm 2\pi k\right) \Rightarrow \sin\theta = -\frac{1}{2}$$

*or*

$$\theta = 330° \pm 360° k\left(\frac{11\pi}{6} \pm 2\pi k\right)$$

$$\Rightarrow \sin\theta = -\frac{1}{2}, \text{ \& etc.}$$

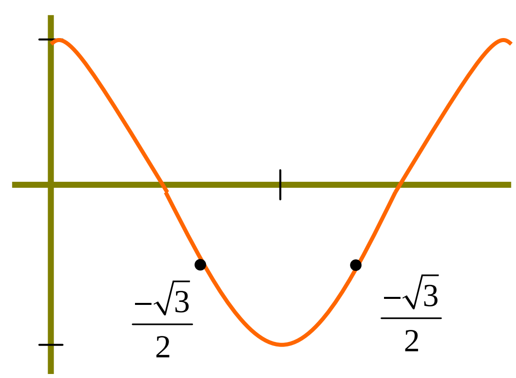
Because all trigonometric functions are periodic from $(-\infty, \infty)$, there will always be an infinite number of solutions (or non-solutions) to any algebraic equation involving a trigonometric expression. Although there are times when numerous answers are needed, for the most part, solutions to trigonometric equations are expected to come from one period of the function's domain. The most common period selected is for $\theta \in [0, 2\pi]$ , which represents one rotation of $\theta$ through the unit circle in the positive (counter-clockwise) direction.

With these restrictions noted, solving trigonometric equations are often accomplished through combinations of trigonometric identities as well. Although identities help to simplify the equation, they may also introduce added restrictions to the equation's solution. For the remainder of this unit, we will introduce many different varieties of trigonometric equations to be solved. It is important for the student to recognize the types of restrictions and techniques being introduced as many equations may be solved in a variety of ways.

*Example #2*: Solve for $\theta \in [0, 2\pi]$ $\qquad 2\cos\theta + \sqrt{3} = 0$

$$2\cos\theta + \sqrt{3} = 0 \quad \Rightarrow \quad \cos\theta = \frac{-\sqrt{3}}{2}$$

Recall the characteristic curve of $y = \cos\theta$



When $\theta = 150°$ or $210° \left(\dfrac{5\pi}{6}, \dfrac{7\pi}{6}\right)$; $\cos\theta = -\dfrac{\sqrt{3}}{2}$

Therefore: $\theta = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ are two solutions to the problem.

*Example #3*: Solve for $\theta \in [0, 2\pi]$ $\sin\theta\cos\theta = \dfrac{1}{4}$

From our list of identities: $\sin 2\theta = 2\sin\theta\cos\theta$

In our problem:

$$\sin\theta\cos\theta = \frac{1}{2}\cdot\frac{1}{2}$$

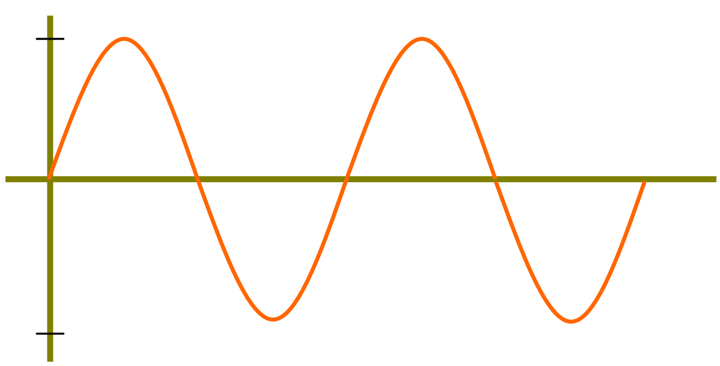
$$2\sin\theta\cos\theta = 2\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$\sin 2\theta = \frac{1}{2}$$

We now need to solve $\sin 2\theta = \dfrac{1}{2}$ for $\theta \in [0, 2\pi]$

We know that for $\sin\theta = \dfrac{1}{2}$, $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

For the expression $\sin 2\theta$ the frequency F = 2, which indicates that the characteristic curve for $f(x) = \sin x$ will repeat twice for $\theta \in [0, 2\pi]$.



This indicates that $\sin 2\theta = \frac{1}{2}$ will have four solutions. To find those values we seek to know when $2\theta = \frac{\pi}{6}$ and $2\theta = \frac{5\pi}{6}$

$$\Rightarrow \qquad \theta = \frac{\pi}{12} \qquad \theta = \frac{5\pi}{12}$$ are the first two solutions.

To find the next two solutions, we look at the period of $\sin 2\theta$. From a previous unit, we know that the frequency (F) and period $(f^{-1})$ are reciprocals.

Therefore, $f^{-1} = \frac{1}{2}$ for $\sin 2\theta$.

Since the usual period of the $\sin\theta$ is $2\pi$, the period involved in this problem is now $2\pi - \frac{1}{2}\cdot(2\pi)$. This value indicates what must be added to each of our fist two solutions in order to find the second set of answers on the interval $[0, 2\pi]$.

For $\frac{\pi}{12} \Rightarrow \frac{\pi}{12} + \pi = \frac{13\pi}{12}$ & For $\frac{5\pi}{12} \Rightarrow \frac{5\pi}{12} + \pi = \frac{17\pi}{12}$

Therefore, for $\sin 2\theta = \frac{1}{2}$, our four solutions are

$$\theta = \frac{\pi}{12}, \quad \frac{5\pi}{12}, \quad \frac{13\pi}{12}, \quad \frac{17\pi}{12}$$

*Example #4*: Solve $\cos^2\theta - \sin^2\theta = 1$ for $\theta$

From our identities: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow \cos 2\theta = 1$

From our identities again: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\cos 2\theta = 1 - 2\sin^2\theta$

$2\cos^2\theta - 1 = 1$ & $1 - \sin^2\theta = 1$

Solving each separately we obtain:

$2\cos^2\theta = 2$ and $1 - 2\sin^2\theta = 1$

$-2\sin^2\theta = 0$

$\cos^2\theta = 1$ $\sin^2\theta = 0$

$\cos\theta = \pm 1$ $\sin\theta = 0$

Both results imply that $\theta \in [0, 2\pi]$.

*Example #5*: Solve $\cos\theta = \cot\theta$

$$\Rightarrow \cos\theta = \frac{\cos\theta}{\sin\theta} \quad \Rightarrow \quad \sin\theta \neq 0 \quad or \quad \theta \neq 0, 2\pi$$

$\sin\theta\cos\theta = \cos\theta$ (cross multiply)

$\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(\sin\theta - 1) = 0$

$\Rightarrow$ ***Case 1*** ***Case2***

$\cos\theta = 0$ $\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\theta = \frac{\pi}{2}$

Therefore $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

*Example #6*: Solve $1+\sin\theta = 2\cos^2\theta$

From our identities; $\cos^2\theta = 1-\sin^2\theta$

$$\Rightarrow \qquad 1+\sin\theta = 2(1-\sin^2\theta)$$

$$1+\sin\theta = 2-2\sin^2\theta$$

$$2\sin^2\theta+\sin\theta-1=0$$

Let $x=\sin\theta \quad \Rightarrow \qquad 2x^2+x-1=0$

Factor $\qquad (2x-1)(x+1) = 0$

| ***Case 1*** | ***Case 2*** |
|---|---|
| $2x-1 = 0$ | $x+1 = 0$ |
| $\Rightarrow \quad 2\sin\theta-1=0$ | $\sin\theta+1=0$ |
| $\sin\theta = \frac{1}{2}$ | $\sin\theta=-1$ |
| $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ | $\theta = \frac{3\pi}{2}$ |

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

*Example #7*: Solve $\sec\theta = \tan\theta + \cot\theta$

$$\Rightarrow \quad \frac{1}{\cos\theta} = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \qquad \Rightarrow \sin\theta \neq 0, \cos\theta \neq 0$$

$$\text{or } \theta \neq 0,\ 2\pi,\ \frac{\pi}{2},\ \frac{3\pi}{2}$$

$$\frac{1}{\cos} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$\frac{1}{\cos\theta} = \frac{1}{\cos\theta\sin\theta}$$

$$\sin\theta\cos\theta = \cos\theta$$

$$\sin\theta\cos\theta - \cos\theta = 0$$

$$\cos\theta(\sin\theta - 1) = 0$$

| ***Case 1*** | ***Case2*** |
|---|---|
| $\cos\theta = 0$ | $\sin\theta - 1 = 0$ |
| $\varnothing$ | $\sin\theta = 1$ |
| | $\theta = \frac{\pi}{2}$ |
| | $\varnothing$ |

Therefore there is no solution to this problem.

**Table of Sines, Cosines, and Tangents**

| Degrees | Radians | cos | sin | tan |
|---|---|---|---|---|
| **0** | $0$ | $1$ | $0$ | $0$ |
| **30** | $\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{3}$ |
| **45** | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| **60** | $\frac{\pi}{3}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\sqrt{3}$ |
| **90** | $\frac{\pi}{2}$ | $0$ | $1$ | undefined |
| **120** | $\frac{2\pi}{3}$ | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $-\sqrt{3}$ |
| **135** | $\frac{3\pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $-1$ |
| **150** | $\frac{5\pi}{6}$ | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| **180** | $\pi$ | $-1$ | $0$ | $0$ |
| **210** | $\frac{7\pi}{6}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $\frac{\sqrt{3}}{3}$ |
| **225** | $\frac{5\pi}{4}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $1$ |
| **240** | $\frac{4\pi}{3}$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $\sqrt{3}$ |
| **270** | $\frac{3\pi}{2}$ | $0$ | $-1$ | undefined |
| **300** | $\frac{5\pi}{3}$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\sqrt{3}$ |
| **315** | $\frac{7\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-1$ |
| **330** | $\frac{11\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| **360** | $2\pi$ | $1$ | $0$ | $0$ |