DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES
LAW OF SINES AND LAW OF COSINES

In a previous unit, we learned how trigonometric identities are used to transform complicated expressions into simpler representations. This unit will expand the list of fundamental identities to include techniques for simplifying expressions that involve angle measures of critical angles. Later in the unit the “Law of Sines” and “Law of Cosines” will be introduced. The laws can be used to find unknown sides or angles in any triangle, not just right triangles.

Double-Angle and Half-Angle Identities

For the first part of this unit, eleven new identities are introduced with examples on how they can be used to simplify trigonometric expressions. Used in combination with the identities introduced in a previous unit, these fundamental identities provide powerful tools for dealing with complex situations involving trigonometry. The following is a list of these new identities.

I.) Double-Angle Identities

1. $\sin 2\theta = 2\sin\theta\cos\theta$

2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

or

3. $\cos 2\theta = 1 - 2\sin^2\theta$

or

4. $\cos 2\theta = 2\cos^2\theta - 1$

5. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

II.) Half-Angle Identities

6. $\sin\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{2}}$

7. $\cos\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1+\cos\alpha}{2}}$

8. $\tan\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$

III.) Miscellaneous Double-Angle Identities

9. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

10. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

11. $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

For the rest of the first part of this unit, examples are provided to show how to incorporate these identities.

Example #1: Show that $\cos^4\theta - \sin^4\theta = \cos 2\theta$

$\cos^4\theta - \sin^4\theta$ can be factored as a difference of squares $\Rightarrow$

$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos 2\theta$

From the "List of Fundamental Trigonometric Identities" in a previous unit, we have $\cos^2\theta + \sin^2\theta = 1$, therefore:

$1 \times (\cos^2\theta - \sin^2\theta) = \cos 2\theta$

which is Identity #2 in this unit. **(verified)**

Example #2: Verify that $\cot(2\theta) = \frac{1}{2}\left(\cot\theta - \tan\theta\right)$

Recall: $\cot\theta = \dfrac{1}{\tan\theta}$

Therefore:

$$\cot 2\theta = \frac{1}{2}\left(\frac{1}{\tan\theta} - \tan\theta\right)$$

$$= \frac{1}{2}\left(\frac{1-\tan^2\theta}{\tan\theta}\right)$$

$$= \frac{1-\tan^2\theta}{2\tan\theta}$$

The last expression is the reciprocal of Identity #5 above and

$\cot 2\theta = \dfrac{1}{\tan 2\theta}$ therefore the identity is **(verified).**

Example #3: Verify that $\sin^2\theta \cdot \cos^2\theta = \frac{1}{8}(1-\cos 4\theta)$

From the list of double-angle identities above:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Therefore we can use our double-angle formulas in the following manner:

$$\sin^2\theta\cos^2\theta = \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{1}{4}(1-\cos 2\theta)(1+\cos 2\theta)$$

$$= \frac{1}{4}(1-\cos^2(2\theta)) \qquad (\text{ by FOIL })$$

From the “List of Fundamental Trigonometric Identities: $1-\cos^2\theta = \sin^2\theta$

$$= \frac{1}{4}(\sin^2(2\theta))$$

From Identity #9 above we have:

$$\frac{1}{4}\left(\frac{1-\cos 4\theta}{2}\right) = \frac{1}{8}\left(1-\cos 4\theta\right) \qquad \textbf{(verified)}$$

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Example #4: Find the exact value of $\cos(22.5°)$

$22.5° = \frac{45°}{2}$ Thus, we can use the Half-Angle Identity #7 from above:

$$\cos\left(\frac{45}{2}\right) = \pm\sqrt{\frac{1+\cos 45°}{2}} \qquad \left(\text{Recall: } 45° = \frac{\pi}{4}\right)$$

$$= \pm\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}}$$

Therefore:

$$\cos(22.5°) = \sqrt{\frac{2+\sqrt{2}}{4}}$$

Example #5: Find the exact value of $\csc\left(\frac{7\pi}{8}\right)$

$$\frac{7\pi}{8} = \frac{1}{2}\cdot\frac{7\pi}{4} \text{ and } \csc\theta = \frac{1}{\sin\theta}$$

Therefore, use the reciprocal of Identity #6 above:

$$\csc\frac{\theta}{2} = \pm\sqrt{\frac{2}{1-\cos\theta}} \Rightarrow \csc\frac{7\pi}{8} = \pm\sqrt{\frac{2}{1-\cos\left(\frac{7\pi}{4}\right)}} \qquad \left(\frac{7\pi}{8} = \frac{\frac{7\pi}{4}}{2}\right)$$

$$= \pm\sqrt{\frac{2}{1-\frac{\sqrt{2}}{2}}}$$

$$= \pm\sqrt{\frac{2}{\frac{2-\sqrt{2}}{2}}}$$

Therefore:

$$\csc\left(\frac{7\pi}{8}\right) = \pm\sqrt{\frac{4}{2-\sqrt{2}}}$$

Example #6: Find the exact value of $\cot 165°$

$2(165) = 330° = \frac{11\pi}{6}$ Therefore, we want to find:

$$\cot\left(\frac{1}{2}\cdot\frac{11\pi}{6}\right) = \cot\left(\frac{11\pi}{12}\right)$$

Use the reciprocal of Identity # 8 above

$$\Rightarrow \cot\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

$$= \pm\sqrt{\frac{1+\cos\frac{11\pi}{6}}{1-\cos\frac{11\pi}{6}}} \qquad \left(\frac{11\pi}{12} = \frac{\frac{11\pi}{6}}{2}\right)$$

$$= \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}} \quad = \pm\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{\frac{2-\sqrt{3}}{2}}}$$

Therefore:

$$\cot(165°) = \pm\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$$

The Law of Sines and the Law of Cosines

So far in our development of trigonometry, we have only examined right triangle trigonometry. However, finding unknown angles and sides in any triangle is an important area of math that applies to real world situations perhaps more frequently than right triangles. In this part of the unit, we will introduce two laws that enable us to find unknown sides and angles in any triangle.

Law of Sines:

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)}$$

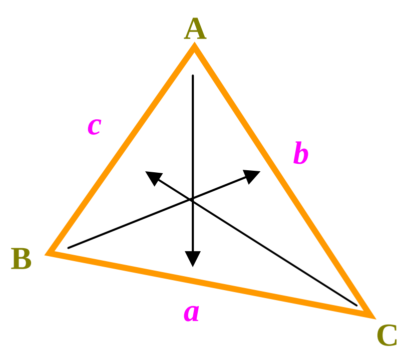
Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(\angle C)$$

$$a^2 = b^2 + c^2 - 2bc\cos(\angle A)$$

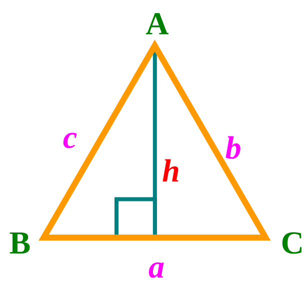
$$b^2 = a^2 + c^2 - 2ac\cos(\angle B)$$

In both these laws, a, b, and c represent the lengths of the sides of the triangle and are opposite the angles $\angle A$, $\angle B$, $\angle C$ as in the diagram below.



Because these two laws provide important tools in using trigonometry to identity real-world values, we will demonstrate how the **Law of Sines** is derived before providing examples how each is used.

Let $\triangle ABC$ be any triangle (we draw this as an acute triangle for simplicity, but the derivation also works for obtuse triangles as well).



From Geometry we draw altitude "h" from $\angle A$ to side $\overline{BC}$ forming two right triangles. From this we obtain:

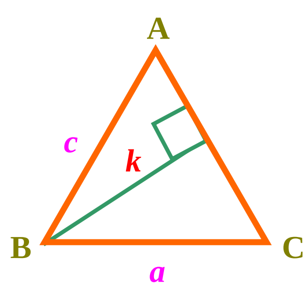
$$\sin\angle B = \frac{h}{c} \quad : \quad \sin\angle C = \frac{h}{b}$$

$$\Rightarrow c\sin\angle B = h \quad : \quad b\sin\angle C = h$$

By substitution $\quad c\sin\angle B = b\sin\angle C$

Then divide $\quad \dfrac{c}{\sin\angle C} = \dfrac{b}{\sin\angle B}$

Next we draw a second altitude from either $\angle B$ or $\angle C$ as in the following (since this altitude is different from **"h"** we label it as altitude **"k"**)



Therefore:

$$\sin\angle A = \frac{k}{c} \quad : \quad \sin\angle C = \frac{k}{a}$$

$$\Rightarrow c\sin\angle A = a\sin\angle C$$

or $\quad \dfrac{c}{\sin\angle C} = \dfrac{a}{\sin\angle A}$

When we combine this with the previous result we have:

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)}$$

The derivation of the Law of Cosines is a bit more complicated and will not be outlined here.

Use the Law of Sines or The Law of Cosines to find the unknown sides or angles in the following problems.

Example #1: For $\triangle ABC$ $\text{m}\angle A = 58°$, $\text{m}\angle C = 38.7°$, c = 1.91

Find $\text{m}\angle B$, *a*, and *b*

a) $\text{m}\angle B = 180° - 58° - 38.7° = 83.3°$

(*Recall from Geometry that the sum of all angles in any triangle* = $180°$)

b) $$\frac{a}{\sin(\angle A)} = \frac{c}{\sin(\angle C)} \Rightarrow \frac{a}{\sin 58°} = \frac{1.91}{\sin 38.7°}$$

(Be sure your calculator is in degree mode and cross multiply)

$$\frac{a}{0.848} = \frac{1.91}{0.625} \Rightarrow 0.625a = 1.91(0.848)$$

$0.625a = 1.6197$

$a = 2.591$

(Answers will be rounded to three decimal places)

c) $$\frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)} \Rightarrow \frac{b}{\sin 83.3°} = \frac{1.91}{\sin 38.7°}$$

(Be sure your calculator is in degree mode and cross multiply)

$$\frac{b}{0.993} = \frac{1.91}{0.625} \Rightarrow 0.625b = 1.91(0.993)$$

$0.625b = 1.8966$

$b = 3.034$

Example #2: m $\angle B = 40.3°$, b = 1.65, c = 2.58

Find m $\angle A$, m $\angle C$, and c

a) $\dfrac{b}{\sin(\angle B)} = \dfrac{c}{\sin(\angle C)} \Rightarrow \dfrac{1.65}{\sin 40.3} = \dfrac{2.58}{\sin \angle C}$

$$\frac{1.65}{0.647} = \frac{2.58}{\sin \angle C}$$

$$2.551 = \frac{2.58}{\sin \angle C}$$

$$\sin \angle C = \frac{2.58}{2.551} = 1.011$$

Since $\sin \angle C$ = 1.011, we can use the arcsine ($\sin^{-1}\theta$) to find the unknown angle measure:

Type $\boxed{\text{2nd}}$, $\boxed{\text{sin}}$, 1.011, $\boxed{\text{ENTER}}$

This results in an error on the calculator, because 1.011 is outside of the range of the $\sin\theta$.

(Recall $R_y : f(\theta) = \sin\theta$)

Does this indicate that there is no solution to the problem? The answer is “no”. But to find this solution, we utilize an identity for the $\sin\theta$ found in a previous unit.

Recall $\dfrac{1}{\sin\theta} = \csc\theta$, therefore, in our original problem we have:

$$\frac{1.65}{\sin(40.3)} = \frac{2.58}{\sin \angle C}$$

$$2.55 = \frac{2.58}{\sin \angle C}$$

$$\frac{2.55}{2.58} = \frac{1}{\sin \angle C}$$

$$\Rightarrow \frac{2.58}{2.55} = \csc \angle C = 1.011$$

Therefore, we want $\csc^{-1}(1.011) = \csc^{-1}(\angle C)$

Type $\boxed{\text{2nd}}$, $\boxed{\sin}$, 1.011, $\boxed{x^{-1}}$, $\boxed{)}$, $\boxed{\text{ENTER}}$

$$\Rightarrow m\angle C = 81.54°$$

b) $\mathrm{m}\angle A = 180 - m\angle B - m\angle C = 180 - 40.3 - 81.54 = 58.16°$

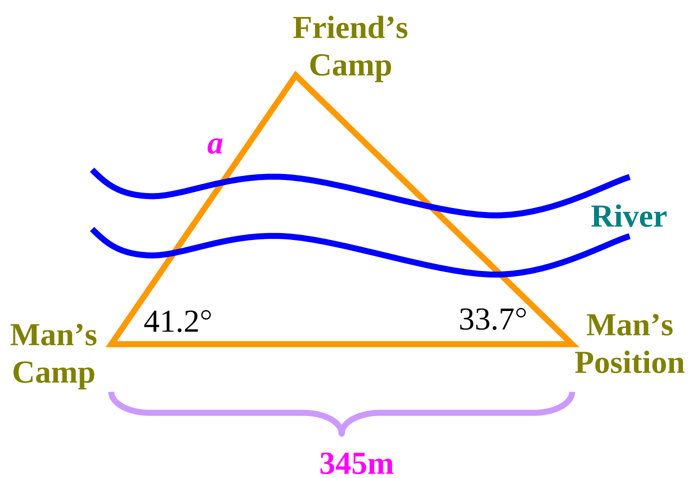
c) $\dfrac{a}{\sin 58.16} = \dfrac{2.58}{\sin 81.54} \Rightarrow a = \dfrac{2.58 \times \sin 58.16°}{\sin 81.54°}$

$$a = 2.216$$

Example #3:

A man stands on one side of a river and is 345m from his camp. On the other side of the river is his friend's camp, which is between the man and his camp. Using a sextant the man measures the angle formed by his camp, himself and his friend's camp as $33.7°$. He then measures the angle formed by his friend's camp, his camp and himself as $41.2°$. How far away is the man's camp from his friend's camp?

Step #1: Draw a triangle to label the situation.



Step #2: Find the measure of the 3rd angle formed with the friend's camp at the vertex

$$180 - 41.2 - 33.7 = 105.1°$$

Step #3: Use Law of Sines to find *a*: $\dfrac{345}{\sin(105.1°)} = \dfrac{a}{\sin(33.7°)} \qquad \Rightarrow a \approx 198\,\text{m}$

Table of Sines, Cosines, and Tangents

Degrees	Radians	cos	sin	tan
0	0	1	0	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	0	1	undefined
120	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
150	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
180	π	-1	0	0
210	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
240	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	0	-1	undefined
300	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
330	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
360	2π	1	0	0