EXPRESSIONS, SEQUENCES, AND LOGIC

This unit is about the order of operations including evaluating expressions with symbols of inclusion. Other topics that will be explored are the properties of real numbers and writing math expressions. In the last sections of this unit, patterns of arithmetic sequences will be explored and the matrix logic table will be investigated to simplify complex logic problems.

Order of Operations

Is the following phrase easy to remember?

“Please excuse my dear Aunt Sally”?

If so, then the order that mathematical operations are completed can also be easily remembered.

P— “**P**lease”—**Parenthesis** are addressed first.
Any operation that appears inside a parenthesis is to be completed before all other operations.

E—“**E**xcuse”—**Exponent**s are simplified next.

M, **D**—“My Dear”— Next in order, **multiplication** and **division** should be completed left to right **in the order that they occur**.

A, **S**—“Aunt Sally”— Finally, complete any **addition** and **subtraction** that remains from left to right **in the order that they occur.**

In expressions that contain more than one operation, it is necessary to follow the order of operations to avoid confusion.

We first identify any symbols of inclusion. Parenthesis ( ) are the most common inclusion symbols; but, others are fraction bars, brackets, [ ], and braces { }.

If more than one symbol is used, we try to find the innermost symbol and begin simplifying there.

Example 1: Simplify $2\left[3\left(4+6\right)\right]$.

$2\left[3\left(4+6\right)\right]$	Start with the inner most parenthesis and add $4+\ 6=10$.
$2\left[3\left(10\right)\right]$	Work inside the brakets next and multiply 3(10) = 30.
$2\left[30\right]$	Now, multiply again.
60	

The expression $2\left[3\left(4+6\right)\right]$ simplifies to 60.

After symbols of inclusion, powers or exponents must be completed next.

Example 2: Simplify $4\left[3\left(3+2\right)^2\right]$.

$4\left[3\left(3+2\right)^2\right]$	Start with the innermost parenthesis and add $3+2=5$.
$4\left[3\left(5\right)^2\right]$	The exponent will be evaluated next, $\left(5\right)^2=25$.
$4\left[3\left(25\right)\right]$	Next, work inside brackets by multiplying $3(25)=75$.
$4\left[75\right]$	Now multiply again.
300	

The expression $4\left[3\left(3+2\right)^2\right]$ simplifies to 300.

Without symbols of inclusion to guide our progress, we should complete multiplication and division from left to right, and then follow with any remaining additions and subtractions working left to right.

Example 3: Simplify $4(5)-18\div 6+2(3)$.

$4(5)-18\div 6+2(3)$	Complete the multiplications and the divsion working from left to right. $4(5)=20$ $18\div 6=3$ $2(3)=6$

Now work the remaining addition and subtraction from left to right.

$20-3+6$	First calculate $20-3=17$.
$17+6$	Then add.
23	

The expression $4(5)-18\div 6+2(3)$ simplifies to 23.

Fraction bars are considered inclusion symbols. All operations should be complete above and below the bar first, and then complete the final step of division.

Example 4: Simplify $\frac{3(2+4)}{2+1}$.

$\frac{3(2+4)}{2+1}$	Start with the innermost parenthesis and add $2+4=6$.
$\frac{3(6)}{3}$	Next, work above the fraction bar; so, multiply 3(6) = 18.
$\frac{18}{3}$	Now divide.
6	

The expression $\frac{3(2+4)}{2+1}$ simplifies to 6.

Math Expressions

Throughout this course and any of the more advanced mathematics courses, it will be necessary to interpret verbal sentences into algebraic sentences. For this, examine the words and phrases that suggest the operation to use.

The chart below lists some of the most common phrases that will be used.

Addition	*Subtraction*	*Multiplication*	*Division*
sum	difference	product	quotient
plus	minus	times	divided by
increased by	decreased by	of	ratio
more than	less than	twice (× 2)	per
total	subtract	multiplied by	average

To translate verbal phrases into algebraic expressions:

a.) chose a variable to represent the unknown (if it is not given)
b.) determine what operation will be used based on the phrase
c.) write the algebraic expression

Example 1: "eight more points than Rachel's score"

a.) Let r represent Rachel's score.

b.) "More than" suggests addition.

c.) The algebraic expression is $r + 8$ or $8 + r$.

Example 2: "four times as much money as Pete"

a.) Let p represent Pete's money.

b.) "Times" suggests multiplication.

c.) The algebraic expression is $p \times 4$, $4 \times p$, or more commonly seen in algebra as $4p$.

Example 3: "twice the quantity of a number plus seven"

a.) Let n represent the number.

b.) "Twice" suggests two times as much.

c.) "Quantity suggests parenthesis $(n + 7)$.

The algebraic expression is $2 \times (n + 7)$, or more commonly seen in algebra as $2(n + 7)$.

*At this point it should be noted that any time a variable or variables are multiplied with a number, it will be written with the number first , and then the variable or variables following. Study the examples given below.

$6x$ means 6 times x OR x times *6.*

$7y$ means 7 times y OR y times *7.*

$3xyz$ means 3 times x times y times z in any order.

Properties of Real Numbers

In this section we will examine the properties of real numbers that will help to solve equations in future units.

Closure Property	
When two real numbers are added, the result is a real number. $5 + 7 = 12$ For any numbers x and y, $x + y = a\ real\ number$	When two real numbers are multiplied, the result is a real number. $7 \times 8 = 56$ For any numbers x and y, $xy = a\ real\ number$

Note: This property may seem to be obvious; but, there are systems of numbers that are not "closed" on some or all operations.

For example, consider the whole numbers, (0, 1, 2, 3, 4, 5…}.

Is the operation of division closed when considering only whole numbers?

Let's test a few pairs of numbers.

Is 6 / 3 a whole number? Yes, because the answer is 2.

Is 6 / 4 a whole number? No, because the answer is 1 1/2 (not a whole number).

Thus, the operation of division is not closed when considering only whole numbers.

* Note: The closure property mentioned in the chart above pertains to all real numbers and is saying that the operations of addition and multiplication are "closed" when considering all real numbers.

<table>
<tr><th colspan="2">Commutative Properties of Addition and Multiplication</th></tr>
<tr><td>The order in which numbers are added does not make a difference in the sum.

$6 + 4 = 4 + 6$

For any numbers x and y,

$x + y = y + x$</td><td>The order in which numbers are multiplied does not make a difference in the product.

$6 \times 4 = 4 \times 6$

For any numbers x and y,

$xy = yx$</td></tr>
</table>

<table>
<tr><th colspan="2">Associative Properties of Addition and Multiplication</th></tr>
<tr><td>The way in which numbers are grouped does not change the sum.

$(2 + 3) + 4 = 2 + (3 + 4)$

For any numbers x, y, and z,

$(x + y) + z = x + (y + z)$</td><td>The way in which numbers are grouped does not change the product.

$(2 \times 3) \times 4 = 2 \times (3 \times 4)$

For any numbers x, y, and z,

$(xy)z = x(yz)$</td></tr>
</table>

Identity Properties of Addition and Multiplication	
The sum of a number and zero is that number. $3 + 0 = 3$ For any number n, $n + 0 = n$	The product of a number and one is that number. $5 \cdot 1 = 5$ For any number n, $n \cdot 1 = n$

Multiplicative Property of Zero
The product of a number and zero is zero. $7 \cdot 0 = 0$ For any number n, $n \cdot 0 = 0$

One of the most important properties in solving equations is the **distributive property.** This property ties addition or subtraction together with multiplication. The distributive property allows the expression to be written in different forms and is given with the following definition.

Distributive Property
The sum or difference of two numbers multiplied by a number is the sum or difference of the product of each number and the number used to multiply. $2(3 + 6) = 2(3) + 2(6) = 6 + 12$ For any number x, y, and z, $x(y + z) = xy + xz$ *The expression $x(y + z)$ is read "x times the quantity of y + z"*

These properties will be helpful when solving equations later in the course. Right now we will identify the properties illustrated by algebraic expressions, so let's practice deciding which property is shown.

Examples:

1) $4 + (9 + 5) = (4 + 9) + 5$ Associative Property of Addition

2) $11+ 14 = 14 + 11$ Commutative Property of Addition

3) $(9 + 2) + 6 = 6 + (9 + 2)$ Commutative Property of Addition

4) $5(0) = 0$ Multiplicative Property of Zero

5) $0 + 17 = 17$ Identity Property of Addition

6) $8(1) = 8$ Identity Property of Multiplication

7) $4(7 + 3) = 4(7) + 4(3)$ Distributive Property

Evaluating Expressions

Algebraic expressions can be **evaluated** by replacing the variable with numbers.

For example, if given the expression $a + b - 24$ and asked to evaluate it for the given values $a = 19$ and $b = 20$, then:

a) Replace the variables, a and b, with the given values, and

b) evaluate the expression using the order of operations.

Let's try the above example.

Example 1: Evaluate $a + b - 24$ if $a = 19$ and $b = 20$.

$a + b - 24$	Given.
$19 + 20 - 24$	Replace a with 19 and b with 20.

Now work the addition and subtraction left to right.

$19 + 20 - 24$	Add $19 + 20 = 39$.
$39 - 24$	Then subtract.
15	

The expression $a + b - 24$ evaluates to 15.

In the next example, two expressions that represent multiplication are used. Recall that $5a$ means 5 times a. Also, the expression bc means b times c.

Example 2: Evaluate $5a + bc - c$ if $a = 4$, $b = 2$, and $c = 3$.

$5a + bc - c$	Given.
$5(4) + (2)(3) - 3$	Replace a with 4, b with 2, and c with 3.
$5(4) + 2(3) - 3$	Work the multiplications from left to right. $5(4) = 20$ $2(3) = 6$
$20 + 6 - 3$	Work the addition and subtraction from left to right. Add $20 + 6 = 26$.
$26 - 3$	Now subtract.
23	

The expression $5a + bc - c$ evaluates to 23.

Example 3: Evaluate $a^2 - \frac{b}{4} + 2c$ if a = 6, b = 4, and c = 5.

$a^2 - \frac{b}{4} + 2c$	Given.
$(6)^2 - \frac{4}{4} + 2(5)$	Replace a with 6, b with 4, and c with 5.
$(6)^2 - \frac{4}{4} + 2(5)$	Start with the exponent since there are no calculations to be completed within parenthesis. $(6)^2 = 6(6) = 36$
$36 - \frac{4}{4} + 2(5)$	Next, work the muliplication and division from left to right. $\frac{4}{4} = 1$ $2(5) = 10$
$36 - 1 + 10$	Now work the addition and subtraction from left to right. Subtract $36 - 1 = 35$
$35 + 10$	Now add.
45	

The expression $a^2 - \frac{b}{4} + 2c$ evaluates to 45.

Sequences

sequence – A sequence is a list of numbers in a certain order connected through a pattern.

term – A term is any of the numbers in a sequence.

arithmetic sequence – An arithmetic sequence is a sequence where any two consecutive terms have a common difference throughout the sequence.

Example 1: Find the next three numbers in this sequence.

{5, 10, 15, 20 ...}

Notice that each number in the sequence is 5 more than the number before it. The common difference is 5.

+5 +5 +5

{5, 10, 15, 20 ...}

To find the next three numbers in the sequence, **follow the pattern** and add five each time.

+5 +5 +5

{5, 10, 15, 20, 25, 30, 35...}

The next three numbers in the sequence are 25, 30, and 35.

Example 2: Find the next three numbers in the arithmetic sequence.

{1.20, 3.46, 5.72, 7.98 ...}

The pattern in this sequence is not as easily seen; BUT, we are given that it is an arithmetic sequence. This means there is a common difference with any two consecutive terms throughout the sequence. So, we'll pick two consecutive terms to find the common difference. We'll pick an additional two consecutive terms just to check.

$5.72 - 3.46 = 2.26$ $\qquad$ $3.46 - 1.20 = 2.26$

We find that each number in the sequence is 2.26 more than the number before it. The common difference is 2.26.

To find the next three numbers in the arithmetic sequence, **follow the pattern** and add 2.26 each time.

+2.26 +2.26 +2.26

{1.20, 3.46, 5.72, 7.98, 10.24, 12.50, 14.76 ...}

The next three numbers in the arithmetic sequence are10.24, 12.50, and14.76.

Matrix Logic

Matrix Logic is a logic or reasoning problem.

A matrix logic table consists of a rectangular array with an appropriate number of columns and rows to record the results of the given clues of a problem. Once all the rows and columns are filled, the conclusion to the complex problem is easier to determine.

Example : Greg, Michael, Krystal, and Anna attend Licking High School. Each student participates in a different school sport. The sports are basketball, football, track, and wrestling. Greg does not participate in basketball or track. Michael does not participate in basketball or football. Krystal prefers one outdoor sport. Name the sport in which each student participates.

What clues are given?

A. Greg does not participate in basketball or track.

B. Michael does not participate in basketball or football.

C. Krystal prefers one outdoor sport. This is an IMPORTANT clue because track is the only *outdoor* sport mentioned.

Now, we are ready to make a matrix table and begin by filling in the easy clue first.

Clue C: We know that Krystal participates in Track … clue … Krystal prefers one *outdoor* sport.

So we put "Yes" in the row beside Krystal's name and under the Track column. We can then put "X's in the row beside Krystal's name for the rest of the sports.

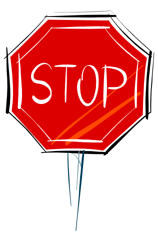
Also, since no one else participates in track, we can put "X's" in the track column for the rest of the students.

	Basketball	Football	Track	Wrestling
Greg			X	
Michael			X	
Krystal	X	X	Yes	X
Anna			X	

Now, we must rely on the table and logic to solve the rest of the problem.

Clue A: Greg does not participate in basketball or track, so we can put an "X" in the row beside Greg's name and in the column under basketball. Note: There is an "X" in the Track column based on the previous clue.

	Basketball	Football	Track	Wrestling
Greg	X		X	
Michael			X	
Krystal	X	X	Yes	X
Anna			X	

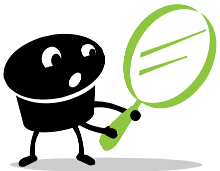


Stop, and take a look at the chart. Can we determine a sport played by another student other than Greg?

Not yet… Two students, Michael or Anna, could play basketball; and there are still three possibilities for each of the other two sports, football and wrestling. We need to add another clue to the chart!

Clue B: Michael does not participate in basketball or football, so we can put "X's" in the row beside Michael's name and in the columns under basketball and football.

	Basketball	Football	Track	Wrestling
Greg	X		X	
Michael	X	X	X	
Krystal	X	X	Yes	X
Anna			X	

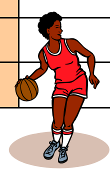


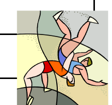
Now, we’ll look at the chart to see if we can determine what sport any of the three remaining students play. There are lots of deductions that can be made now!

Michael has “X’s” in all sports except Wresting, so he is the wrestler.
For basketball, the only student left is Anna, so she must be the basketball player.

Now, let’s fill in the chart to see what sport is left for Greg.

	Basketball	Football	Track	Wrestling
Greg	X		X	
Michael	X	X	X	Yes
Krystal	X	X	Yes	X
Anna	Yes	X	X	X





Greg is not the wrestler because that is Michael’s sport; thus, Greg plays football.

Let’s complete the chart.

	Basketball	Football	Track	Wrestling
Greg	X	Yes	X	X
Michael	X	X	X	Yes
Krystal	X	X	Yes	X
Anna	Yes	X	X	X

With the help of matrix logic, we know that Greg plays football, Michael participates in wrestling, Krystal participates in track, and Anna plays basketball.