SOLVING QUADRATIC EQUATIONS; COMPLEX NUMBERS

In this unit you will solve quadratic equations using the Quadratic formula. You will also use the discriminant of the quadratic formula to determine how many and what type of solutions the quadratic equation will have. The unit will conclude with operations on complex numbers.

The Quadratic Formula

The quadratic formula is used to solve any quadratic equation in standard form, $ax^2 + bx + c = 0$. The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the quadratic formula

1.) make sure the equation is in standard form
2.) label the values of *a*, *b*, and *c*
3.) replace the values into the equation and solve

Example #1: Use the quadratic formula to solve the given quadratic for "*x*".

$x^2 - 16x - 36 = 0$ $\qquad$ $a = 1$, $b = -16$, $c = -36$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(-36)}}{2(1)}$$

$$x = \frac{16 \pm \sqrt{256 + 144}}{2}$$

$$x = \frac{16 \pm \sqrt{400}}{2}$$

$$x = \frac{16 \pm 20}{2}$$

$$x = \frac{16 + 20}{2} \quad \text{and} \quad x = \frac{16 - 20}{2}$$

$$x = \frac{36}{2} \quad \text{and} \quad x = \frac{-4}{2}$$

$$x = 18 \quad \text{and} \quad x = -2$$

Example #2: Use the quadratic formula to solve the given quadratic for "*x*".

$x^2 + 4x - 18 = 0$ $\qquad$ $a = 1$, $b = 4$, $c = -18$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-18)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 72}}{2}$$

$$x = \frac{-4 \pm \sqrt{88}}{2}$$

$$x = \frac{-4 + \sqrt{88}}{2}, \; x = \frac{-4 - \sqrt{88}}{2}$$

*These expressions can be simplified and this will be addressed in a later unit.

*If a quadratic function is in standard form, $ax^2 + bx + c = y$, then it is possible to locate the axis of symmetry by using the following formula:

Axis of Symmetry: $\quad x = \dfrac{-b}{2a}$

Example #3: Find the axis of symmetry for the given quadratic.

$f(x) = 2x^2 + 8x + 19$ $\qquad$ $a = 2$, $b = 8$, $c = 19$

The axis of symmetry is $x = \dfrac{-8}{2(2)} \rightarrow x = -2$

The axis of symmetry also refers to the *x*-value of the vertex.

To find the *y*-value of the vertex:

1.) replace the value of *x* into the equation
2.) solve for *y*

Example #4: Find the vertex of the parabola of the given quadratic.

$y = 2x - 2 + x^2$ — a = 1, b = 2, c = –2

$y = x^2 + 2x - 2$ — -put in standard form

-find the axis of symmetry

$x = \frac{-2}{2(1)}$ — -the axis of symmetry is x = –1

-replace all x values with –1 and solve for y

$$y = (-1)^2 + 2(-1) - 2$$
$$y = 1 - 2 - 2$$
$$y = -3$$

Therefore the vertex of this parabola will be located at (–1, –3).

Imaginary Numbers

Up to this point in your mathematics classes you have been told that there is no such thing as the square root of a negative number. Well, now we are going to talk about imaginary numbers. The rule for imaginary numbers is as follows:

$\sqrt{-1} = i$	$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = i^0 = 1$

To use the quadratic formula in the next lesson, you will have to be familiar with $\sqrt{-1} = i$. We will come back to the other powers of i later.

If $r > 0$, then the **imaginary number** $\sqrt{-r}$ is defined as the following

$$\sqrt{-r} = \sqrt{-1} \cdot \sqrt{r} = i\sqrt{r}$$

If you are trying to find the square root of a negative number, the first thing you need to do is factor out a (–1) which is equal to *i*.

Example #1: Simplify: $\sqrt{-16}$

$= \sqrt{-1} \cdot \sqrt{16}$

$= i \cdot 4$

$= 4i$ *always put the number value first

We will always put the whole number first, and then the i for the imaginary part. If the simplified form does not contain a whole number, then the i will be first. Look at the example below.

Example #2: Simplify: $\sqrt{-5}$

$= \sqrt{-1} \cdot \sqrt{5}$

$= i\sqrt{5}$

Finding the solution to a quadratic that contains an imaginary number is done as follows:

Example #3: Find the solution to the following quadratic: $6x^2 - 3x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a = 6, *b* = –3, and *c* = 1

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(1)}}{2(6)}$$

$$x = \frac{3 \pm \sqrt{9 - 24}}{12}$$

$$x = \frac{3 \pm \sqrt{-15}}{12}$$

$$x = \frac{3 \pm i\sqrt{15}}{12}$$

$$x = \frac{3 + i\sqrt{15}}{12} \quad \text{and} \quad x = \frac{3 - i\sqrt{15}}{12}$$

Note: The final answer may also be written as:

$$x = \frac{3}{12} \pm \frac{i\sqrt{15}}{12}$$

$$x = \frac{1}{4} + \frac{i\sqrt{15}}{12} \quad \text{and} \quad \frac{1}{4} - \frac{i\sqrt{15}}{12}$$

The Discriminant

In the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is known as the discriminant and will identify how many and what type of solutions there are to a quadratic equation.

Types of Solutions

If the value of $b^2 - 4ac$ is **positive**	2 real solutions
If the value of $b^2 - 4ac$ is **negative**	2 imaginary solutions
If the value of $b^2 - 4ac$ is **zero**	1 real solution

In a previous topic about the Quadratic Formula, we found that there are two real solutions (18 and –2) for the quadratic equation $x^2 - 16x - 36 = 0$. Let's take a look at the value of the discriminant and determine if there are indeed two real solutions.

Example #1:

$x^2 - 16x - 36 = 0$ $a = 1$, $b = -16$, $c = -36$

discriminant $b^2 - 4ac$

replace *a*, *b*, and *c* $(-16)^2 - 4(1)(-36)$

$256 - (-144)$

$= 400$ Since this is a positive number, this means that there will be two (2) real solutions.

This verifies that indeed our solution to the quadratic has two roots.

Example #2: How many and what type of solution(s) will $2x^2 - 4x + 3 = 0$ have?

1.) $a = 2$, $b = -4$, $c = 3$

2.) replace each of these numbers into the discriminant $b^2 - 4ac$

3.) $(-4)^2 - 4(2)(3)$ -simplify

4.) $16 - 24 = -8$

5.) since this is a negative number, this means that the solution to this equation will be two (2) imaginary numbers.

Let's take a look at solving this equation using the quadratic formula.

1.) replace *a*, *b*, and *c* into the quadratic formula $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2.) $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$

3.) simplify $x = \dfrac{4 \pm \sqrt{16 - 24}}{4}$

$$x = \frac{4 \pm \sqrt{-8}}{4}$$

4.) remember we have to factor out the $\sqrt{-1}$ and turn this into i

$x = \dfrac{4 \pm i\sqrt{8}}{4}$ we will leave our answer like this until we learn how to simplify radical expressions.

Example #3: How many and what type of solution(s) will $x^2 - 4x + 4 = 0$ have?

1.) $a = 1$, $b = -4$, $c = 4$

2.) replace each of these numbers into the discriminant $b^2 - 4ac$

3.) $(-4)^2 - 4(1)(4)$ -simplify

4.) $16 - 16 = 0$

5.) Since this answer is equal to zero it means that the solution to this quadratic is 1 real solution.

Let's take a look at solving this equation using the quadratic formula.

1.) replace *a*, *b*, and *c* into the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2.) $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)}$

3.) simplify $x = \frac{4 \pm \sqrt{16 - 16}}{2}$

$x = \frac{4 \pm \sqrt{0}}{2}$

$x = \frac{4 + 0}{2}$ $x = \frac{4 - 0}{2}$

$x = 2$

Example #4: Find the discriminant and determine the number of solutions for each of the quadratics shown below.

1.) $3x^2 - 6x + 4 = 0$	2.) $4x^2 - 20x + 25 = 0$	3.) $9x^2 + 12x = -2$
$b^2 - 4ac$	$b^2 - 4ac$	$b^2 - 4ac$
$(-6)^2 - 4(3)(4)$	$(-20)^2 - 4(4)(25)$	$(12)^2 - 4(9)(2)$
$36 - 48 = -12$	$400 - 400 = 0$	$144 - 72 = 72$
2 imaginary solutions	1 real solution	2 real solutions

*Note: In the third quadratic equation, express the quadratic equation in standard form, $9x^2 + 12x + 2 = 0$, to determine a = 9, b = 12, and c = 2.

Complex Numbers

A **complex number** is any number that can be written as $a+bi$, where a and b are real numbers and $i=\sqrt{-1}$, a is called the **real part** and b is called the **imaginary part**.

*Two complex numbers are equal if the real parts are equal and the imaginary parts are equal.

Example #1: Solve for "x" and "y": $-3x+4iy=21-16i$

real parts	**imaginary parts**
$-3x=21$	$4iy=-16i$
$x=-7$	$y=-4$

Thus $x=-7$ and $y=-4$

Adding and Subtracting Complex Numbers

To add or subtract complex numbers

-combine the real parts

-combine the imaginary parts

Example #1: Find the sum: $(-10-6i)+(8-i)$

$$(-10+8)+(-6i-i)$$

$$-2-7i$$

Example #2: Find the difference: $(-9+2i)-(3-4i)$

$$(-9-3)+(2i-(-4i))$$

$$-12+6i$$

Multiplying Complex Numbers

Powers of *i*

Remember in the previous unit that it was stated we would use the powers of i in this unit? Well, this is where we need to review the powers of *i*. You will need to know this pattern as we multiply complex numbers.

The powers of *i* are cyclic and repeat in a pattern of “four” numbers.

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = i^0 = 1$

To multiply complex numbers

-use FOIL multiplication
-combine like terms
-change i^2 to (–1)

Example #1: Find the product: $(2-i)(-3-4i)$

$$-6-8i+3i+4i^2$$
$$-6-5i+4(-1)$$
$$-6-5i-4$$
$$-10-5i$$