PROBABILITY: FUNDAMENTAL COUNTING PRINCIPLE, PERMUTATIONS, COMBINATIONS

In this unit you will begin by learning the fundamental counting principle and applying it to probabilities. You will then explore permutations, which are used when the outcomes of the event(s) depend on order, and combinations, which are used when order is not important.

Introduction to Probability

Probability is the likelihood of an event occurring.

Terminology (a coin is used for each of the examples)	
Definition	**Example**
Trial: a systematic opportunity for an event to occur	tossing a coin in the air
Experiment: one or more trials	tossing a coin 6 times
Sample space: the set of all possible outcomes of an event	H or T
Event: an individual outcome or any specified combination of outcomes.	landing H or landing T

Probability is expressed as a number from 0 to 1. It is written as a fraction, decimal, or percent.

- an impossible event has a probability of 0
- an event that must occur has a probability of 1
- the sum of the probabilities of all outcomes in a sample space is 1

The probability of an event can be assigned in two ways:

1.) **experimentally**: approximated by performing trials and recording the ratio of the number of occurrences of the event to the number of trials. (as the number of trials in an experiment increases, the approximation of the experimental probability increases).

2.) **theoretically:** based on the assumption that all outcomes in the sample space occur randomly.

Theoretical Probability

If all outcomes in a sample space are equally likely, then the theoretical probability of event *B*, denoted *P*(*B*), is defined by:

$$P(B) = \frac{\text{number of outcomes in event B (favorable outcomes)}}{\text{number of outcomes in the sample space (possible outcomes)}}$$

Example #1: Find the probability of randomly selecting an orange marble out of a jar containing 3 blue, 3 red, and 2 orange marbles.

$$P(\text{1 orange}) = \frac{\text{favorable}}{\text{possible}} = \frac{\text{2 orange}}{\text{8 possible}}$$

$$= \frac{2}{8} = \frac{1}{4} \text{ or } 25\%$$

Fundamental Counting Principle

If there are *m* ways that one event can occur and *n* ways that another event can occur, then there are ***m*** x ***n*** ways that both events can occur.

Example #2: Emily is choosing a password for access to the Internet. She decides not to use the digit 0 or the letters A, E, I, O, or U. Each letter or number may be used more than once. How many passwords of 3 letters followed by 2 digits are possible?

Use the fundamental counting principle. There are 21 possible letters and 9 possible digits.

1st letter		2nd letter		3rd letter		1st digit		2nd digit
21	×	21	×	21	×	9	×	9

The number of possible passwords for Emily is $21^3 \cdot 9^2$ or 750,141.

Permutations

A **permutation** is an arrangement of objects in a specific order. When objects are arranged in a row, the permutation is called a **linear permutation**.

Permutations of *n* Objects

The number of permutations of n objects is given by $n!$ (! is called factorial and means to multiply all consecutive natural numbers starting with n).

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example #1: On a baseball team, nine players are designated as the starting line up. Before a game, the coach announces the order in which the nine players will bat. How many different orders are possible?

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 362,880 \text{ possible orders.}$$

*Note: When the coach is choosing, on his first choice he has nine players to choose from. Once he makes that choice, he the has eight players left to choose from, then seven, then six, and so on.

If you want to use your calculator to find 9! Press 9, $\boxed{\text{MATH}}$, move the cursor over to PRB, and go down to 4:! Then press $\boxed{\text{ENTER}}$.

Permutations of *n* Objects Taken *r* at a Time

The number of permutations of n objects taken r at a time, denoted by $P(n, r)$ or ${}_nP_r$ is given by:

$$P(n,r) = {}_nP_r = \frac{n!}{(n-r)!}, \text{ where } r \le n$$

Example #2: Find the number of ways to listen to 6 different CD's from a selection of 18 CD's.

$$ {}_{18}P_6 = \frac{18!}{(18-6)!} $$

$$ = \frac{18!}{12!} = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12!} $$

$$ = 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 $$

$$ = 13{,}366{,}080 $$

Note: Since the order in which the CD's will be played **is** important, this is a "permutation" problem.

Calculate the Number of Possible Outcomes

As we try to calculate probability, we find that each situation may be slightly different. For instance, if we considered randomly selecting letters from a word and the word we chose had repeated letters, we would not get a clear picture of the probability.

Example #1: What is the probability of selecting the letter "r" from the letters in the word random?

r-a-n-d-o-m

1-r 6-letters total probability $= \frac{1}{6}$

Example #2: What is the probability of selecting the letter "s" from the word success?

3-s 7-letters total probability $= \frac{3}{7}$

There is a higher probability when there are more chances of success.

When considering the arrangement of letters, use permutations. Before we look at permutations, we need to understand factorial numbers. Let's take a look.

n! is read "*n* factorial"

6! means $6 \times 5 \times 4 \times 3 \times 2 \times 1$ which is a product of all the natural numbers, starting with 6 and going down to 1.

6! = 720

Example #3: Evaluate 10!

10! means $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$10! = 3{,}628{,}800$

If you want to use your calculator to find 10! Press10, $\boxed{\text{MATH}}$, move the cursor over to PRB, and go down to 4:! Then press $\boxed{\text{ENTER}}$.

Now we are ready for permutations.

Example #4: How many ways can the letters of the word "random" be arranged?

This example requires a permutation. It's formula is $P(n,n) = n!$ where we are selecting all of the letters in the arrangement.

$$P(n,n) = n!$$
$$P(6,6) = 6!$$
$$P(6,6) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$P(6,6) = 720$$

There are 720 ways the letters in the word "random" may be arranged.

Example 5: If we look at arranging letters in the word "success", we need to realize that when an s or c is selected, it does not matter which is which. So there are less ways to select the arrangement.

This is called a permutation with repetition and is given by the following formula $P = \frac{n!}{a!b!}$ where "*a*" and "*b*" are repeating letters.

How many ways are there to arrange the letters in the word success?

We are using all 7 letters but the "s" has 3 repeats and the "c" has 2 repeats.

$$P = \frac{7!}{3!2!}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4}^{2} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot 2}{1}$$

$$P = 420$$

The letters in the word "success" may be arranged 420 different ways.

Combinations

An arrangement of objects in which order is **not** important is called a combination.

Combinations of *n* Objects Taken *r* at a Time

The number of combinations of *n* objects taken *r* at a time is given by:

$$C(n,r) = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ where } 0 \le r \le n$$

$C(n,r)$, ${}_nC_r$, and $\binom{n}{r}$ have the same meaning.

All are read "n choose r".

Example #1: How many ways are there to give 4 honorable mention awards to a group of 10 students?

$${}_{10}C_4 = \frac{10!}{4!(10-4)!}$$

$$= \frac{10!}{4!(6!)} = \frac{10\cdot 9\cdot 8\cdot 7\cdot (6!)}{4!(6!)}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7\cdot (\cancel{6!})}{4!(\cancel{6!})} \qquad \text{-cancel the } 6!\text{'s}$$

$$= \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}$$

$$= 210$$

Note: Since the order in which the honorable mention awards are presented is **not** important, then this is a “combination” problem.