TRIGONOMETRIC IDENTITIES

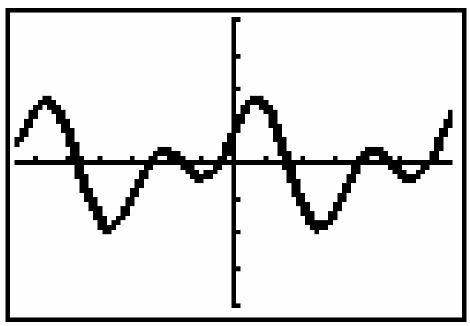
In this unit you will look closely at trigonometry ratios and identities. You will begin with examining the usefulness of trigonometry using a graphing calculator. You will then look at the development and derivations of trigonometric identities. In the final part of the unit, you will verify trigonometric identities.

Introduction to Trigonometric Ratios and Identities

Trigonometry is a useful tool in that the trigonometric functions can be used to describe many assorted real-world situations that have a tendency to repeat or are "periodic". However, trigonometric equations that describe these situations can easily become quite complicated and involved. Consider the following expression and graph the equation on your calculator.

$$Y_1 = 2\sin x\cos x + \cos x$$

Now enter $Y_2 = \sin(2x) + \cos x$ and press your left arrow to the far left of the $\boxed{\text{Y=}}$ editor. Press $\boxed{\text{ENTER}}$ to thicken the graphed line before graphing. Graph the 2nd equation after thickening the line.



You will notice that both equations graph the same curve, but the equation entered into Y_2 can be considered a simpler form of the two equations because it involves entering less terms. If we set Y_1=Y_2 and subtract cos x from both sides, we obtain the following:

$$Y_1 = Y_2$$

$$2\sin x\cos x + \cos x = \sin(2x) + \cos x$$

$$2\sin x\cos x = \sin(2x)$$

The last line in this equation represents a statement of a "Trigonometric Identity". Because the two sides of the equation are equal, the right side of the equation may be substituted for any expression that involves the left side to simplify the expression, such as in the following:

$$y = \frac{\sin(2x)}{\cos x} = \frac{2\sin x\cos x}{\cos x} = 2\sin x$$

with a suitable restriction on $\cos x \neq 0$ which would result in division by zero in the original problem.

In this unit we will develop a general understanding of how trigonometric identities are developed and perform exercises that utilize these identities to simplify complex trigonometric expressions. Before beginning, it is important to make two observations on the previous example.

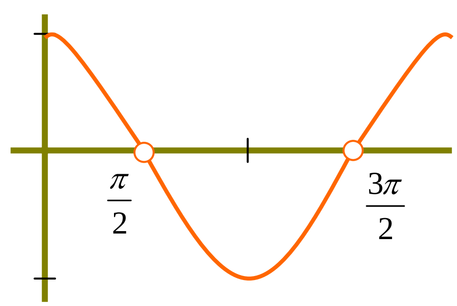
1.) Since the original equation stated that $y = \frac{\sin 2x}{\cos x}$, all subsequent simplifications of this expression must account for the restriction that cos $x \neq 0$. Although the expression eventually simplified to y = 2sin x, which, as we have previously learned, is the characteristic curve for the sin x with an amplitude (A= 2), the original expression takes precedence. Since cos x = 0 when $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, these values must be excluded from the domain of the simplified expression, y = 2sin x. This is important to note because both forms of the expression can be entered and graphed on the calculator, but the calculator will not indicate any difference between the two curves at these restricted values. To see this for yourself, enter the two equations into $\boxed{\text{Y=}}$ and graph each separately.

$$\text{Y}_1 = 2\sin x \ \boxed{\text{Graph}}$$

$$\text{Y}_2 = \frac{\sin 2x}{\cos x} \ \boxed{\text{Graph}}$$

***Be sure to turn off Y_1 before viewing Y_2 by moving the cursor left to highlight the equals sign, and then press enter.**

Although both graphs are identical, $\text{Y}_2 = \frac{\sin 2x}{\cos x}$ clearly involves division by zero, and therefore, there should be a hole in the graph at $x = \frac{\pi}{2}, \frac{3\pi}{2}$ such as:



We will return to this concept in a later unit when we solve trigonometric equations for a variable. For this unit we will only mention these restrictions when significant to the problem.

2.) The second item to note and emphasize is that $2\sin x \neq \sin(2x)$.

Recall from a previous unit that the standard equation of a trigonometric function is $f(x) = A\sin(Bx + C)$ where A= amplitude and B= frequency of the curve. It is important to note the distinction between the two values so that the following type of incorrect simplification does not occur as we proceed through the unit:

Simplify: $y = \dfrac{2\sin x}{\sin 2x}$

A.) CORRECT simplification:

$$\frac{2\sin x}{\sin 2x} = \frac{2\sin x}{2\sin x \cos x} = \frac{1}{\cos x} = \sec x$$

B.) INCORRECT simplification:

$$\frac{2\sin x}{\sin 2x} = \frac{2\sin x}{2\sin x} = 1$$

or also INCORRECT:

$$\frac{2\sin x}{\sin 2x} = \frac{2\sin x}{\sin x \sin x} = \frac{2}{\sin x}$$

As we proceed through this unit, remember that for situations such as $2\sin x$: A= 2, F=1 and that for $\sin 2x$: A= 1, F= 2 which are not equal.

Development of Trigonometric Identities

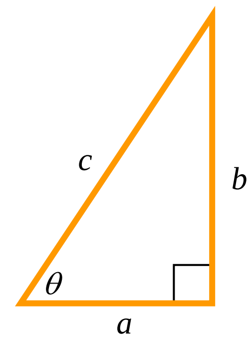
Because trigonometric identities allow for the simplification of complex expressions and also because there are literally hundreds of these identities, it is instructive to gain an intuitive understanding of how these identities are derived from the six(6) basic ratio relationships and the use of inscribed triangles on the unit circle.

Recall from a previous unit that we defined the six(6) ratios in the following manner:

$$\sin\theta = \frac{b}{c} \qquad \csc\theta = \frac{c}{b}$$

$$\cos\theta = \frac{a}{c} \qquad \sec\theta = \frac{c}{a}$$

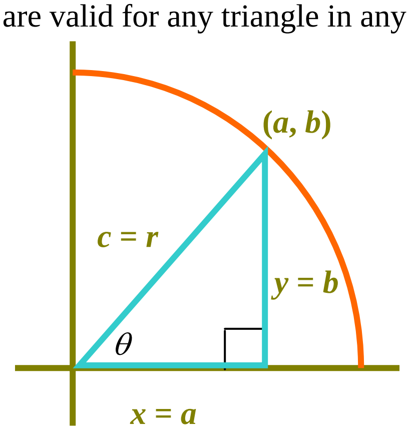
$$\tan\theta = \frac{b}{a} \qquad \cot\theta = \frac{a}{b}$$



As noted, the reciprocals of the first three ratios, $\sin\theta,\ \cos\theta,\ \tan\theta$, define the remaining three ratios $\csc\theta,\ \sec\theta,\ \cot\theta$. From this reciprocal relationship, we obtain the first three in our list of trigonometric identities which can be used for simplification purposes.

$$\sin\theta = \frac{1}{\csc\theta} : \quad \cos\theta = \frac{1}{\sec\theta} : \quad \tan\theta = \frac{1}{\cot\theta}$$

To understand the development of the remaining identities, it is instructive to return to the graph of the unit circle with inscribed triangles. (Note: Although we view only quadrant #I, the results obtained are valid for any triangle in any quadrant.)



Because the trigonometric ratios uniquely define a relationship between angles and sides in any right triangle, we will allow $c = r$ to represent any length of the hypotenuse of a right triangle or any radius of the unit circle.

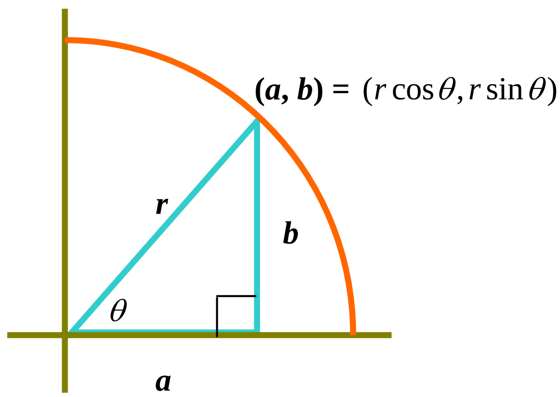
From the definitions of our six(6) ratios, we also know that:

$$\sin\theta = \frac{y}{r} \qquad\qquad \cos\theta = \frac{x}{r}$$

Multiply each side by "*r*" and we obtain our next identity:

$$y = r\sin\theta \qquad x = r\cos\theta$$

These are called the "coordinate identities" and are used in graphing points in **Polar Coordinates** which we will examine in a later unit. For now, we mention that in our previous graph, the point (a, b) now becomes:



For the next identity, we use our coordinate identities and recall the Pythagorean Theorem.

$a^2 + b^2 = c^2$ which for our diagram becomes; $x^2 + y^2 = r^2$

By substitution:

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$r^2\cos^2\theta + r^2\sin^2\theta = r^2$ (Note: $\sin\theta \cdot \sin\theta = \sin^2\theta$ is common notation as opposed to $\sin\theta \cdot \sin\theta = \sin\theta^2$)

Dividing by r^2, the next trigonometric identity is:

$$\sin^2\theta + \cos^2\theta = 1$$

For the next identity, we return to our six(6) fundamental ratios. Recall:

$$\tan\theta = \frac{b}{a}, \quad \sin\theta = \frac{b}{c}, \quad \cos\theta = \frac{a}{c}:$$

$$\text{now} \quad \frac{b}{c} \div \frac{a}{c} = \frac{b}{c} \cdot \frac{c}{a} = \frac{b}{a} = \tan\theta$$

$$\text{and} \qquad \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{\sin\theta}{\cos\theta}$$

Therefore:

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \quad \& \quad \frac{\cos\theta}{\sin\theta} = \cot\theta$$

For the last derivation, we use the first six identities and find:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \Rightarrow \qquad \frac{1}{\csc^2\theta} + \frac{1}{\sec^2\theta} = 1$$

The common denominator of the right side = $\csc^2\theta\sec^2\theta$, forming the single fraction:

$$\frac{\sec^2\theta + \csc^2\theta}{\csc^2\theta\sec^2\theta} = 1$$

Cross multiply and obtain:

$$\sec^2\theta + \csc^2\theta = \csc^2\theta\sec^2\theta$$

Through repeated similar operations on the above identities, other fundamental trigonometric identities can be established which, when combined with those already obtained, yields even more results, and so on.

List of Fundamental Trigonometric Identities

For this unit, the following list of common identities will be used to verify other identities and evaluate certain results.

I.) Single Angle Identities:

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\Rightarrow 1 - \sin^2\theta = \cos^2\theta$$
$$\Rightarrow 1 - \cos^2\theta = \sin^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin\theta = \cos(90 - \theta)$$

&

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

II.) Sum and Difference Identities:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

&

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Verification of Trigonometric Identities

Example #1: Verify the following Identity: $\sin\theta(\cot\theta+\tan\theta)=\sec\theta$

To verify an identity, it is important that only one side of the equation be manipulated. Either the left side must be transformed into the right side or the right side transformed into the left side. Cross multiplication is NOT allowed in verification problems, BUT can be used to establish a new identity.

Step #1: Use: $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$ $\Rightarrow$ $\sin\theta\left(\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}\right)=\sec\theta$

Step #2: Distribute:

$$\cos\theta+\frac{\sin^2\theta}{\cos\theta}=\sec\theta$$

Step #3: Common Denominator of left side = $\cos\theta$

$$\frac{\cos^2\theta}{\cos\theta}+\frac{\sin^2\theta}{\cos\theta}=\sec\theta \quad\Rightarrow\quad \frac{\cos^2\theta+\sin^2\theta}{\cos\theta}=\sec\theta$$

Step #4: Use: $\sin^2\theta+\cos^2\theta=1$

$$\frac{\cos^2\theta+\sin^2\theta}{\cos\theta}=\sec\theta \quad\Rightarrow\quad \frac{1}{\cos\theta}=\sec\theta$$ **(Verified)**

Example #2: Verify the following identity: $1-\dfrac{\cos^2\theta}{1+\sin\theta}=\sin\theta$

Step #1: Use: $\cos^2\theta = 1-\sin^2\theta$

$$1-\frac{1-\sin^2\theta}{1+\sin\theta}=\sin\theta$$

Step #2: Factor $1-\sin^2\theta$ as difference of squares and cancel:

$$1-\frac{(1-\sin\theta)\cancel{(1+\sin\theta)}}{\cancel{1+\sin\theta}}=\sin\theta \quad\Rightarrow\quad 1-1+\sin\theta=\sin\theta$$

$\sin\theta=\sin\theta$ **(Verified)**

Example #3: Verify the following identity: $\dfrac{\sin\theta\cdot\cos\theta}{\cos^2\theta-\sin^2\theta}=\dfrac{\tan\theta}{1-\tan^2\theta}$

Step #1: Use: $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

$$\frac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{\dfrac{\sin\theta}{\cos\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}}$$

Step #2: Common denominator of $1-\dfrac{\sin^2\theta}{\cos^2\theta}$ is $\cos^2\theta$

Therefore:

$$\frac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{\dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$$

Step #3: Invert and multiply the right side, and then cancel:

$$\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}=\frac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$ **(Verified)**

Example #4: Verify the following identity: $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

Step #1: The common denominator of the left =

$$(1-\sin\theta)(1+\sin\theta)=(1-\sin^2\theta)=\cos^2\theta$$

Therefore: $\dfrac{1+\sin\theta}{\cos^2\theta}+\dfrac{1-\sin\theta}{\cos^2\theta}=2\sec^2\theta$

&

$$\frac{1+\sin\theta+1-\sin\theta}{\cos^2\theta}=\frac{2}{\cos^2\theta}=2\sec^2\theta$$

$2\sec^2\theta=2\sec^2\theta$ **(Verified)**

Example #5: Verify the following Identity: $\sin\left(\dfrac{\pi}{2}+\theta\right)=\cos\theta$

Step #1: Use: $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$\sin\left(\frac{\pi}{2}+\theta\right)=\sin\frac{\pi}{2}\cos\theta+\cos\frac{\pi}{2}\sin\theta=\cos\theta$$

Step #2: We know $\sin\dfrac{\pi}{2}=1$: $\cos\dfrac{\pi}{2}=0$

Therefore: $1\cdot\cos\theta+\cancel{0\cdot\sin\theta}=\cos\theta$

$\cos\theta=\cos\theta$ **(Verified)**

Example #6: Verify the following Identity: $\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\tan\alpha+\tan\beta$

Step #1: Use: $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}=\tan\alpha+\tan\beta$$

Step #2: Separate into two fractions and cancel:

$$\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}=\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}=\tan\alpha+\tan\beta$$

Step #3: Use: $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

$$\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}=\tan\alpha+\tan\beta$$ **(Verified)**

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Example #7: Verify the following identity: $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$

Step #1: Use: $\cot\theta=\dfrac{1}{\tan\theta}$ on the left side:

$$\cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}$$

Step #2: Use $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$\frac{1}{\tan(\alpha+\beta)}=\frac{1}{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

Step #3: Use $\tan\theta=\dfrac{1}{\cot\theta}$ again $\Rightarrow\dfrac{1-\dfrac{1}{\cot\alpha\cot\beta}}{\dfrac{1}{\cot\alpha}+\dfrac{1}{\cot\beta}}$

Step #4: Find the common denominator of the numerator and denominator of the main fraction.

$$\frac{1-\dfrac{1}{\cot\alpha\cot\beta}}{\dfrac{1}{\cot\alpha}+\dfrac{1}{\cot\beta}}=\frac{\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha\cot\beta}}{\dfrac{\cot\beta+\cot\alpha}{\cot\alpha\cot\beta}}=\frac{\cot\alpha\cot\beta-1}{\cancel{\cot\alpha\cot\beta}}\cdot\frac{\cancel{\cot\alpha\cot\beta}}{\cot\beta+\cot\alpha}$$

$$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$$ **(Verified)**

Example #8: Verify $\cos 75° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ is the exact value.

$$\cos 75° = \cos(30° + 45°) = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$$

Use: $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$ **(Verified)**

Example #9: Find $\sin 105°$

$$\sin 105° = \sin(60 + 45) = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$ **(Verified)**

Example #10: Find $\tan 15°$

$$\tan 15° = \tan(45-30) = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$\tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}} = \frac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}} == \frac{3-\sqrt{3}}{3}\cdot\frac{3}{3+\sqrt{3}}$$

Multiply by conjugate: $= \frac{3-\sqrt{3}}{3+\sqrt{3}} = \frac{(3-\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3})(3-\sqrt{3})} = \frac{9-6\sqrt{3}+3}{9-3}$

$$= \frac{12-6\sqrt{3}}{6} = 2-\sqrt{3}$$ **(Verified)**

Example #11: Find $\sec(195°)$

Step #1: Use: $\sec\theta = \dfrac{1}{\cos\theta}$

Therefore: $\sec(195) = \dfrac{1}{\cos(195)} = \dfrac{1}{\cos(45+150)}$

Step #2: Use $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$\cos(45+150) = \cos\left(\frac{\pi}{4}+\frac{5\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{5\pi}{6} - \sin\frac{\pi}{4}\sin\frac{5\pi}{6}$$

$$= \frac{\sqrt{2}}{2}\left(\frac{-\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{-\sqrt{6}-\sqrt{2}}{4}$$

Step #3: Find reciprocal of result in *Step #2.*

$$\frac{1}{\dfrac{-\sqrt{6}-\sqrt{2}}{4}}$$

$$= \frac{-4}{\sqrt{6}+\sqrt{2}}\cdot\frac{(\sqrt{6}-\sqrt{2})}{(\sqrt{6}-\sqrt{2})}$$

$$= \frac{4\sqrt{2}-4\sqrt{6}}{6-2} = \frac{4(\sqrt{2}-\sqrt{6})}{4} = \sqrt{2}-\sqrt{6}$$ **(Verified)**