INVERSE TRIGONOMETRIC FUNCTIONS

In previous units, we have made the following observations:

An equation of the form $y = \sin^{-1} x$ is not to be regarded as the reciprocal of the $y = \sin x$.

The reciprocal of the $y = \sin x$, or any trigonometric function would be more correctly noted as $y = (\sin x)^{-1}$.

Instead, trigonometric functions of the form, $y = \sin^{-1} x$, $y = \cos^{-1} x$, etc. are called the "arcsine", "arccosine", etc.

In addition, the "arctangent" was used to find an unknown angle measure given the ratio of two sides of a right triangle. For an equation of the form, $y = \tan x$, the equation represents the ratio of two sides of the triangle given a known angle measure, x.

By examining these two situations together, it is implied that the functions

$$f(x) = \sin x \ \& \ f^{-1}(x) = \sin^{-1} x$$
$$f(x) = \cos x \ \& \ f^{-1}(x) = \cos^{-1} x$$
$$f(x) = \tan x \ \& \ f^{-1}(x) = \tan^{-1} x$$

are inverse functions in the following manner:

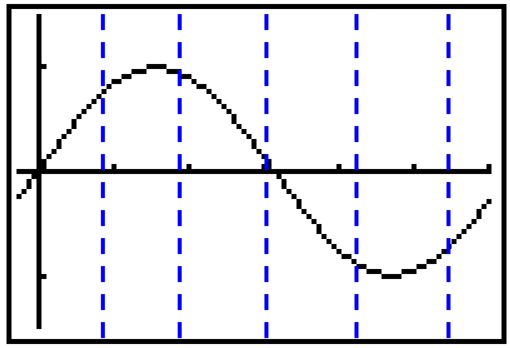
(a) For $y = \sin x, y = \cos x$, and $y = \tan x, y$ is the ratio of two sides of a right triangle given the value of an angle x.

(b) For $y = \sin^{-1} x$, $y = \cos^{-1}$, and $y = \tan^{-1}$, y is the value of an angle given the ratio of two sides of a right triangle.

In this unit we will examine the relationships that exist between the trigonometric functions and their inverses, how they are identified, and develop graphing techniques to find their characteristic curves.

Review of Inverse Functions

Each characteristic curve of all six trigonometric ratios are functions since any vertical line will intersect each graph at only one point on the curve (Vertical Line Test: VLT)

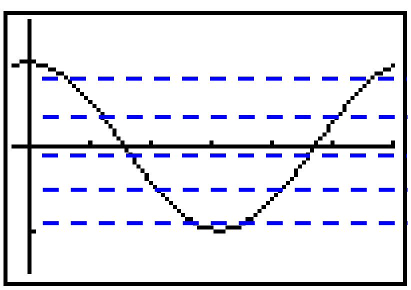


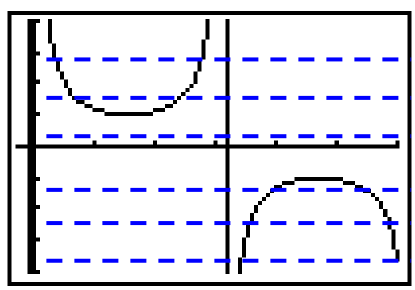
The above diagram demonstrates the VLT for $y = \sin x$. All remaining five trigonometric curves also pass this same test.

We now recall another important concept , the identification of a function as a "one-to-one" (1-1) function.

A function is "1-1" if ALL domain values of the original function become the range values of the inverse of the function and ALL range values of the function become the domain values of the inverse. To determine if a function is "1-1" we use the Horizontal Line Test: HLT. In the HLT, a function is "1-1" if a horizontal line intersects the curve only once throughout the range (y-values) of the curve. Clearly, no trigonometric curve is a "1-1" function, as none of the six characteristic curves passes the HLT.

Example #1:





The above diagrams demonstrate the HLT for $y = \cos x$ and $y = \csc x$. Similar results can be seen for the remaining four trigonometric ratios.

From the above discussion, no trigonometric curve has a true “1-1” inverse for all values of their domains and ranges. In order to find inverses for the six trigonometric ratios that are also “1-1” functions, it will be necessary to restrict the Domain and Range of each function.

Restricting Trigonometric Curves to Find their Inverse Functions

In order to find a "1-1" inverse function for any trigonometric curve, it is necessary to restrict the domain and range of the characteristic curve so that the function is "1-1". The restrictions on these curves must be accomplished in such a way that the inverse function reflects all critical values of the original curve.

The Inverse of *f(x)* = sin *x*

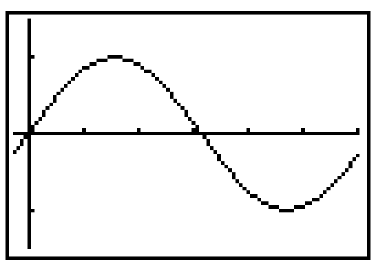
The characteristic curve of $y = \sin x$ indicates that the function's range oscillates between -1 and 1 throughout its Domain. Because this and all trigonometric ratios oscillate between certain values, we say that the six trigonometric ratios are "periodic".

From the definition of the Inverse of a function, we know that the domain of the original function becomes the range of the inverse, and that the range of the original becomes the domain of the inverse.

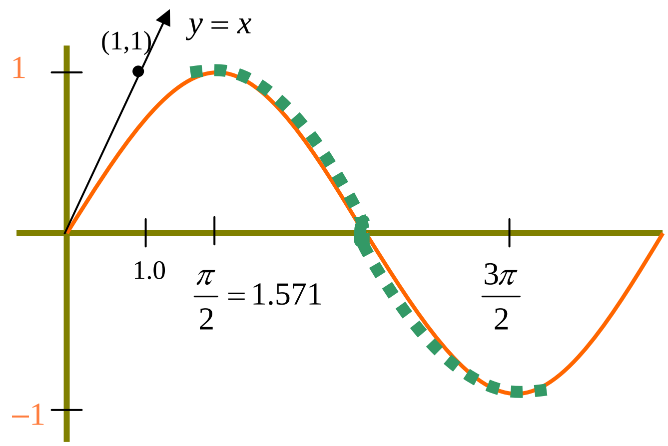
For $f(x) = \sin x$, $R_y : f(x) = [-1,1]$,

Therefore: $D_x : f^{-1}(x) = \sin^{-1} x \Rightarrow [-1,1]$

In order to define the restricted Domain of $f(x) = \sin x$ that will reflect all critical values of the function in the inverse, we again examine the characteristic curve of $y = \sin x$.



For $f(x) = \sin x$, $D_x : f(x) = [0, 2\pi]$. However, we want to restrict this interval so that this is a "1-1" function. From the characteristic curve, we can restrict the domain of $f(x)$ to the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and the range values of $f(x)$ take on all values for the interval [-1,1] as is shown in the following diagram.



$$f(x) = \sin x, \quad \text{for} \qquad D_x : f(x) = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$
$$R_y : f(x) = [-1, 1]$$

Although this set of restrictions causes *f*(*x*) = sin *x* to be "1-1", and therefore, $y = \sin^{-1} x$ to be the inverse function, this form of restriction has a number of draw backs.

(1) When inverse functions were first discussed in this course, one of the main concepts introduced was that of symmetry. Recall that the line $y = x$ was found to be the line of symmetry between a function and its inverse. For the restrictions listed above, the line $y = x$ is graphed. Although, if graphed, the inverse of the restricted function would be above the line $y = x$, the symmetry about this line would be disconnected. In addition, symmetry about the critical value of (0, 0) is not readily apparent.

(2) Recall that trigonometric functions define a unique relationship between the sides of a right triangle and the *acute* angles in these triangles. For the restrictions that we have outlined above, we have identified angles on the interval from $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ = $[90°, 270°]$. When the values for the trigonometric ratios were developed using the unit circle and inscribed triangles, the values were derived by using acute reference angles in each right triangle. By restricting f(*x*) = sin *x* to a domain interval of $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the significance of these reference angles is overlooked. In order to define a more consistent

restriction on the Domain of $f(x)$, and thus the Range of $f^{-1}(x)$, we need to expand our examination of the characteristic curve for *y* = sin *x*.

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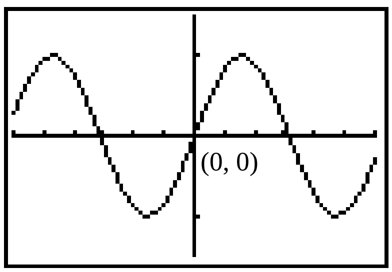
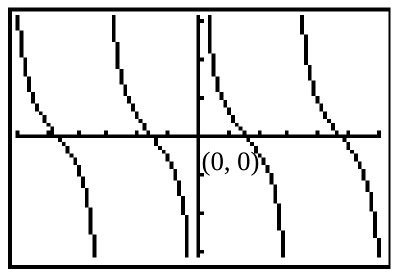
When defining the values for *y* = sin *x* from the unit circle, a counterclockwise rotation of the radius defined positive angle measures from 0 to 2π. If the radius is rotated clockwise, the angle measures become negative. Because the angle measures formed constitute the *x*-values for the graph of the characteristic curves, the six trigonometric ratios can be extended to negative *x*-values. In addition, the rotation angle may also exceed $-2\pi = -360°$ and $2\pi = 360°$. In this way, the $D_x : f(x) = \sin x$ => $(-\infty, \infty)$ and the characteristic curve for this, and all trigonometric ratios, periodically repeat from $(-\infty, \infty)$. The curves, *y* = sin *x* and *y* = cot *x*, are shown below as illustrations.

$y = \sin x$

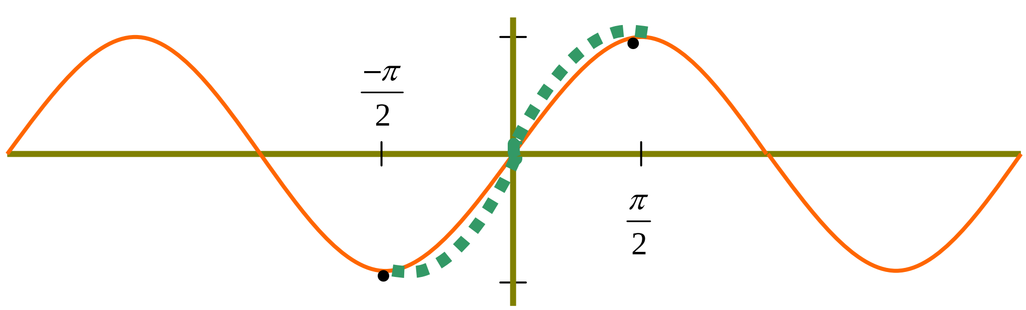


$y = \cot x$



From this, a better restriction on the Domain of *f*(*x*) = sin *x* becomes

$D_x : f(x) = \sin x$ => $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and is outlined on the graph below.



In the above diagram, the Domain restrictions reflect the acute reference angles for inscribed triangles in the unit circle and the range still retains all values from [-1, 1]. From this discussion, we obtain the following definitions for *f*(*x*) = sin *x* as a "1-1" function and for $f^{-1}(x) = \sin^{-1} x$.

Let $f(x) = \sin x$: with $D_x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ $R_y \in [-1, 1]$, Then $f^{-1}(x) = \sin^{-1} x$ for $D_x \in [-1,1]$ $R_y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Using the Graphing Calculator to View Inverse Trigonometric Functions

Although tabular values for points on $y = \sin^{-1} x$ may be calculated and plotted, this can be timeconsuming. In order to view $y = \sin^{-1} x$, type the following into your calculator.

[Y=], [2*nd*], [sin], x (Note: Be sure you are in "Radian" [MODE])

Based on the intervals defined above, input the following window settings, and then [GRAPH] .

Xmin = -1.2

Xmax = 1.2

Ymin = $\frac{-\pi}{2} - 0.2$

Ymax = $\frac{\pi}{2} + 0.2$

If you now enter the following into Y_2 & Y_3 ,the symmetrical relationships between $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$ can be seen.

$Y_2 = (\sin(x))(-\pi/2 \le x)(x \le \pi/2)$
$Y_3 = x$

*Recall that the commands "$\le$", and, "and" are found by pressing [2nd] [MATH]

The Inverse of f(x) =sec x

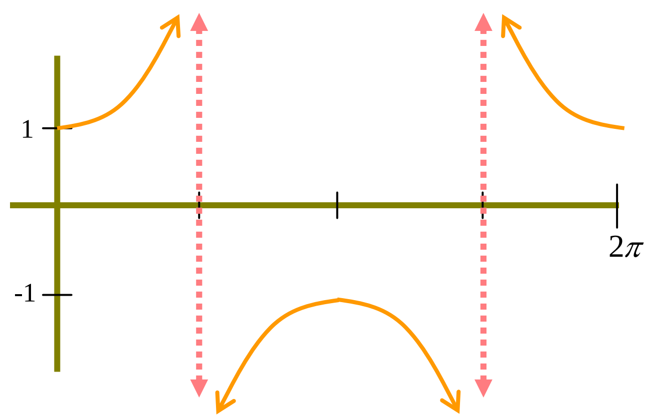
From our previous examination for $y = \sin^{-1} x$ and $f^{-1}(x)$, the concept of symmetry played a key role along with the fact that the inverse functions of trigonometric functions should reflect the critical values found in each ratio. In addition, the values of the acute angles of the inscribed right triangles of the unit circle also make a contribution in defining the inverse of a trigonometric ratio. However, for the remaining five trigonometric ratios, not all of these factors can be considered with the same importance in order to define the inverse of the function. In the assignment for this unit, you will be asked to define the restrictions on $f(x) = \tan x$, $f(x) = \csc x$, $f(x) = \cos x$ and $f(x) = \cot x$ in order to make each a "1-1" function. You will also define and graph the inverse of each of these functions based on your restrictions. In performing this task, the following guidelines will assist you in determining the nature of the inverses for these trigonometric ratios:

1.) Identify the limits of the $R_y : f(x)$ first.
2.) Identify an angular interval for $D_x : f(x)$ nearest to 0^r that takes on all the range values of $f(x)$.
3.) Incorporate the significance of the acute reference angles for the inscribed right triangles in the unit circle, *where possible*.

Let us examine how these guidelines can be used to define the inverse function $f^{-1}(x) = \sec^{-1} x$.

Step #1: Examine the characteristic curve of $y = \sec x$ for both positive and negative angles.

The characteristic curve for $y = \sec x$ is shown below from $[0, 2\pi]$



From this graph, we can set a window on the graphing calculator to view the graph into negative angle measures.

Press WINDOW and set the window to:

Xmin = $-2\pi - 0.2$

Xmax = $2\pi + 0.2$

From the Characteristic Curve we see that the

$$R_y : f(x) = \sec x \Rightarrow (-\infty, -1] \cup [1, \infty)$$

and from this interval we can set the following:

Ymin = -4

Ymax = 4

(Note: The choice of “Ymin” and “Ymax” can be any value sufficiently large enoughto view the nature of the curve’s range values.)

Step #2: Press [Y=] :

Recall that $f(x) = \sec x$ is the reciprocal of $f(x) = \cos x$, but not the inverse. Since the graphing calculator does not have keys for $\csc x, \sec x$ and $\cot x$, we utilize the reciprocal nature of these values in order to view the graph of $y = \sec x$ by entering the following:

$Y_1 = (\cos(x))^{-1}$ and press [GRAPH]

Note: As mentioned at the beginning of this unit, this is not the same as entering $Y_1 = \cos^{-1}(x)$ which is the inverse of the cosine function, for the calculator.

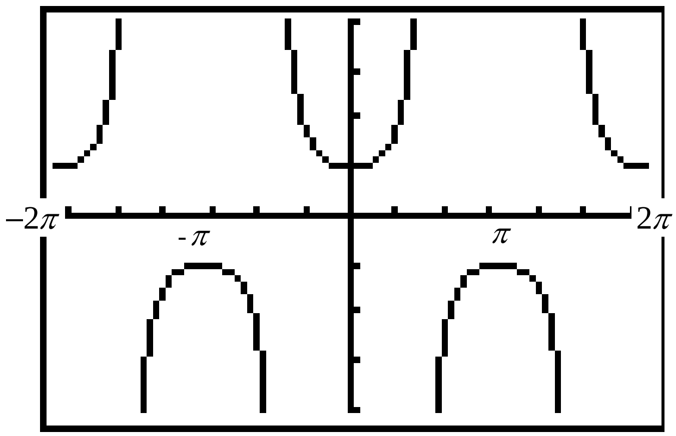
Study the chart below to see the correct interpretation of trig functions and their inverses; and, note how each must be entered into the graphing calculator.

Trig Function	**Graphing Calculator’s Entry for the Trig Function**	**Notation for the Inverse of the Function**	**Graphing Calculator’s Entry for the Inverse Function**
$\sin x$	$\sin x$	$\sin^{-1} x$	$\sin^{-1} x$
$\cos x$	$\cos x$	$\cos^{-1} x$	$\cos^{-1} x$
$\tan x$	$\tan x$	$\tan^{-1} x$	$\tan^{-1} x$
$\sec x$	$(\cos(x))^{-1}$	$\sec^{-1} x$	$\cos^{-1}(1/x)$
$\csc x$	$(\sin(x))^{-1}$	$\csc^{-1} x$	$\sin^{-1}(1/x)$
$\cot x$	$(\tan(x))^{-1}$	$\cot^{-1} x$	$\pi/2 - \tan^{-1}(x)$ *

*Note: $y = \cot^{-1} x$ is stated in a different manner due to the restrictions on the range of $y = \tan^{-1} x$.

The following extended characteristic curve is now displayed for $y = \sec x$.

$y = \sec x$



Step #3: Define $D_x : f^{-1}(x) = \sec^{-1} x$ and $R_y : f^{-1}(x) = \sec^{-1} x$. As we already know, $y = \sec x$ is not "1-1" and must be restricted. From the graph above, the range values used to display the graph can be used as the domain values of $f(x) = \sec^{-1} x$ or:

$$D_x : f^{-1}(x) = \sec^{-1} x \Rightarrow (-\infty, -1] \cup [1, \infty)$$

However the $R_y : f^{-1}(x) = \sec x$ comes from the domain $D_x : f(x) = \sec x$. In finding the inverse for $f(x) = \sin x$, it was possible to restrict the domain on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. This restriction incorporated the significance of the acute reference angles for the inscribed triangles in the unit circle used to evaluate the six trigonometric ratios. However, these same domain restrictions cannot be used for $f(x) = \sec^{-1} x$ as this would not restrict the graph to be "1-1". We must then choose an adjusted domain for the function that makes the ratio "1-1", but still reflects all critical values in the function.

According to guideline #2 above, this restricted Domain should be chosen as close to 0^r as possible. If we examine the interval $[0, \ \pi]$, we see that this interval incorporates all possible range values for $y = \sec x$. (Although the interval $[-\pi, \ 0]$ also accomplishes this task, it is preferable that the inverse reflect positive rotation angles for the trigonometric function over negative angles.) Although negative angles were chosen in defining $f^{-1}(x) = \sin x^{-1}$, this was done to better reflect the symmetry of the two functions.

For $y = \sec x$ and its inverse, this form of symmetry about 0^r does not exist. Therefore choosing positive rotation angles constitutes the best choice for the $R_y : f^{-1}(x) = \sec^{-1} x$.

From this analysis, we obtain the following definitions for the restricted function $f(x) = \sec x$ and the inverse $f^{-1}(x) = \sec^{-1} x$

$$f(x) = \sec x \quad \text{for} \quad D_x : f(x) = [0, \pi], x \neq \frac{\pi}{2}, \quad R_y : f(x) = (-\infty, -1] \cup [1, \infty)$$

and

$$f^{-1}(x) = \sec^{-1} x \quad \text{for} \quad D_x : f^{-1}(x) = (-\infty, -1] \cup [1, \infty), \quad R_y : f^{-1}(x) = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$$

Step #4: Graph $f(x) = \sec x$, $f^{-1}(x) = \sec^{-1} x$, $f(x) = x$ as restricted "1-1" functions.

Press Y= and type the following

$f(x) = \sec(x) \rightarrow Y_1 = (\cos(x))^{-1}(-0.2 \leq x)(x \leq \pi + 0.2)$

$f(x) = \sec^{-1}(x) \rightarrow Y_2 = (\cos^{-1}(1/x))$

$f(x) = x \rightarrow Y_3 = x$

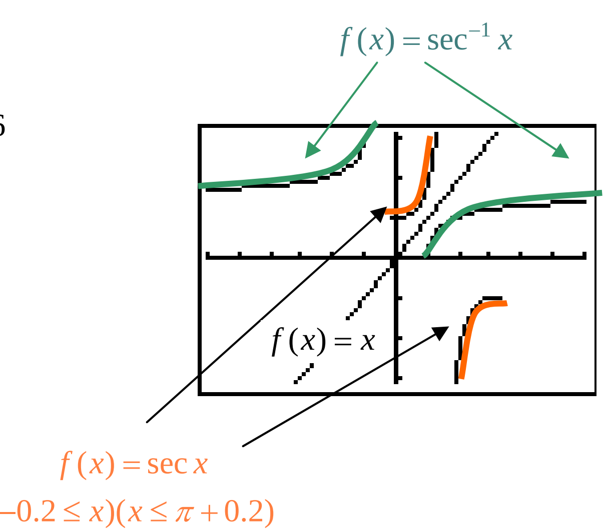
For WINDOW : Use the Domain and Range below:

Xmin = -6

Xmax = 6

Ymin = -6

Ymax = 6



Notice that the calculator's graph of $y = \sec^{-1} x$ is limited. This is because the calculator is programmed to define inverse trigonometric values for specific angles and specific conditions. For this course we will define the inverse trigonometric functions as outlined in this unit. As you advance your studies in higher mathematics more specific restrictions will become necessary.