

GRAPHING TRIGONOMETRIC FUNCTIONS

In a previous unit, we examined how to evaluate the six trigonometric ratios from a right triangle inscribed in a unit circle. Specifically, seventeen critical angle values were examined and the values of the sine, cosine, tangent, etc. were evaluated for these angles. In this unit, we will plot those results on a graph and define a characteristic curve for each of the six relationships. Later we will examine the standard equation of a trigonometric curve and adjust the components of that equation to vary the shape of the standard graphs.

Graphing the Characteristic Curves of the Six Trigonometric Functions

The Standard Form of a Trigonometric Equation (Function)

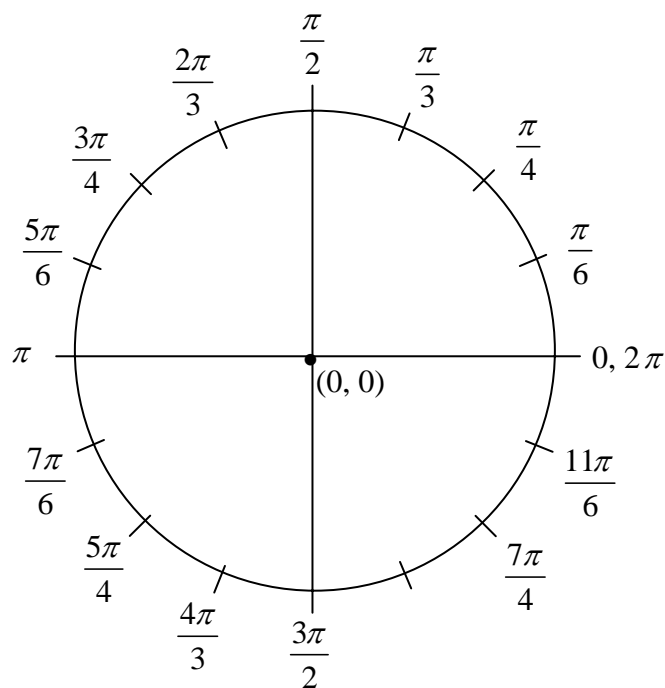
Graphing Trigonometric Functions for all Values of A, F, and P in the Standard Equation

Graphing the Characteristic Curves of the Six Trigonometric Functions

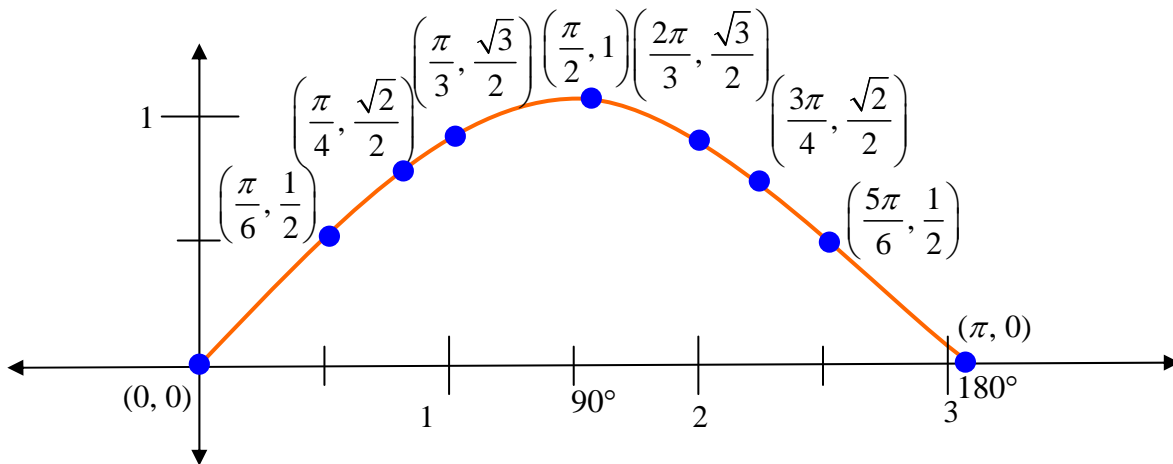
In a previous assignment, we found the trigonometric ratios for seventeen critical angle measures from 0^r to $2\pi^r$ (0° to 360°), 102 values in all. In each instance, the value of the angle determined the position of the radius (hypotenuse) and the values of the legs of the inscribed right triangle in the unit circle. In this sense, the angle values are the independent variable and the resulting trigonometric ratio is the dependent variable. The ratios obtained for the values of the sine θ are shown below in tabular form.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin \theta$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0



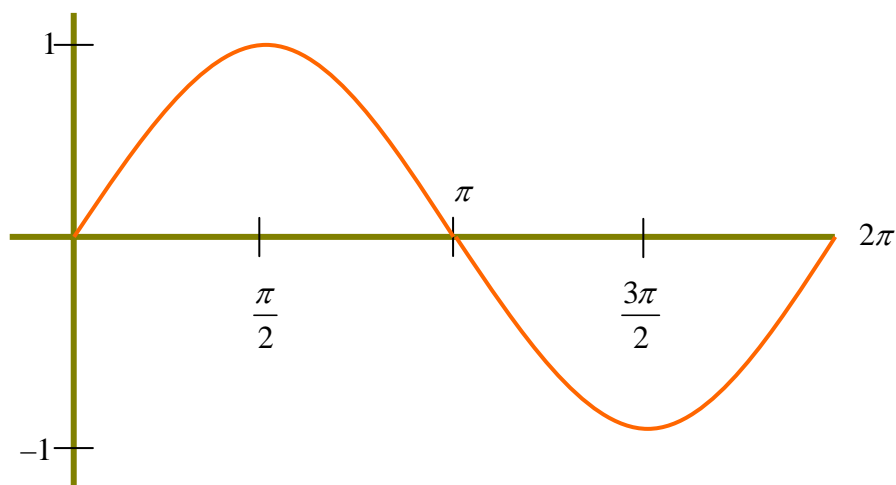
Each of these values can be paired and graphed using points on the xy -plane by allowing “ θ ” to be placed on the x -axis and the $\sin \theta$ to be graphed on the y -axis in the following manner:



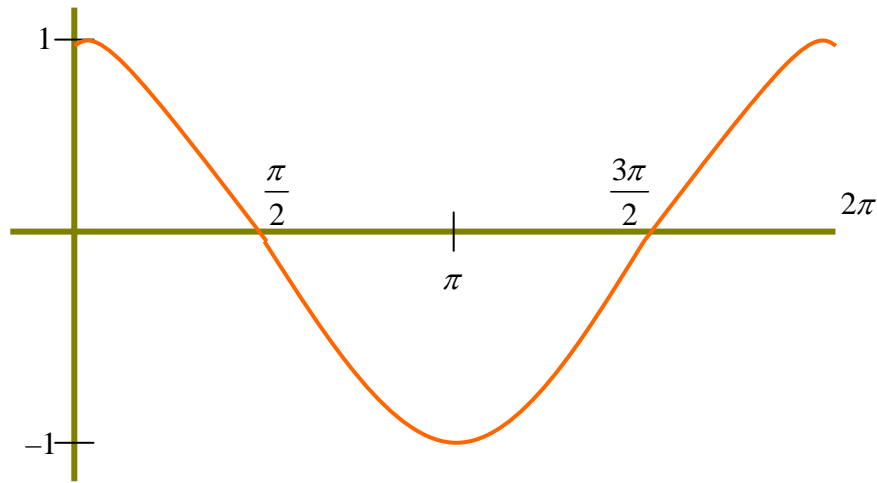
Note: $\frac{\pi}{6} = 0.524$, $\frac{\pi}{4} = 0.785$, $\frac{\pi}{3} = 1.047$, $\frac{\pi}{2} = 1.571$, and so on...

Note: $\frac{1}{2} = 0.5$, $\frac{\sqrt{2}}{2} = 0.707$, $\frac{\sqrt{3}}{2} = 0.866$

If all angles and resulting ratios for the $\sin \theta$ are also graphed in this manner the following standard graph of the $\sin \theta$ is obtained:



If we repeat this process for the $\cos \theta$, the following characteristic curve for this value is also obtained:



At this point, a problem occurs in attempting to graph the remaining four trigonometric ratios. For each of the remaining four ratios, **undefined values** occur at several critical angles measures. To graph these four remaining curves, we must first interpret the meaning of the undefined values for a graph on the xy -plane.

We begin by recalling that, when graphing linear equations, a line with undefined slope graphs a vertical line on the plane. From this, it is reasonable to conclude that for the undefined values found in the $\tan \theta$, $\csc \theta$, $\sec \theta$, and $\cot \theta$, a vertical line can be drawn on the graph. The next question to consider is whether or not this vertical line is actually a part of the graph of each of these curves. Let us examine the tabular values for the $\tan \theta$ to answer this question.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\tan \theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

Notice that on each side at the two undefined values of the $\tan \theta$, (at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$), the sign of the $\tan \theta$ changes from positive to negative or negative to positive. To understand the meaning of this sign change we can employ the use of the graphing calculator.

On the calculator press **MODE** and select “Degree” so that we can interpret our results in the commonly used degree measure. Recall that the values for our six trigonometric functions were found only from the seventeen critical values on the unit circle but that there are an infinite number of angle values found from 0° to 360° (0^r to $2\pi^r$). We convert the radian measures of the critical values on each side of the undefined values of the $\tan \theta$ and obtain the following:

$$\frac{\pi}{3} = 60^\circ \qquad \frac{\pi}{2} = 90^\circ \qquad \frac{2\pi}{3} = 120^\circ$$

$$\frac{4\pi}{3} = 240^\circ \qquad \frac{3\pi}{2} = 270^\circ \qquad \frac{5\pi}{3} = 300^\circ$$

Obviously, there are an infinite number of angle measures for which we could set up an inscribed triangle in the unit circle and evaluate the $\tan \theta$. Fortunately, the calculator can calculate enough of these values so that we should gain an understanding of the graphs for the remaining four trigonometric functions on either side of the undefined values.

Enter the following values on the graphing calculator to record the value of the $\tan \theta$ on the left side of $90^\circ \left(\frac{\pi}{2}\right)$ as the angle values get closer to 90° .

On the Home screen press:

$$\boxed{\tan}, 70^\circ, \boxed{\text{ENTER}} = 2.747$$

$$\boxed{\tan}, 80^\circ, \boxed{\text{ENTER}} = 5.671$$

$$\boxed{\tan}, 85^\circ, \boxed{\text{ENTER}} = 11.430$$

$$\boxed{\tan}, 87^\circ, \boxed{\text{ENTER}} = 19.081$$

$$\boxed{\tan}, 89^\circ, \boxed{\text{ENTER}} = 57.290$$

$$\boxed{\tan}, 89.5^\circ, \boxed{\text{ENTER}} = 114.589$$

$$\boxed{\tan}, 89.75^\circ, \boxed{\text{ENTER}} = 229.182$$

Continue entering values of the $\tan \theta$ for degree measures less than 90° but closer to 90° to see that as the angle measure approaches 90° the $\tan \theta$ approaches positive infinity.

Next enter the following values for the $\tan \theta$ to the right of 90° in the order shown below to see that as the value of the angle gets closer to 90° from the right, the value of $\tan \theta$ approaches negative infinity.

$$\tan 100^\circ = -5.57$$

$$\tan 95^\circ = -11.430$$

$$\tan 93^\circ = -19.081$$

$$\tan 91^\circ = -57.290$$

$$\tan 90.5^\circ = -114.589$$

$$\tan 90.25^\circ = -229.182$$

(Similar results can also be obtained for all remaining undefined values not only for the $\tan \theta$ but for $\csc \theta$, $\sec \theta$, and $\cot \theta$).

We can now conclude that the vertical line placed on the graph for the undefined values of these four curves is not part of the graph of the curve but acts as a boundary (or “asymptote”) between the intervals where the values of the curves change sign. With the aid of the graphing calculator, we can obtain a general view for the remaining four curves and their characteristic graphs. Since all angle values so far calculated have been positive (0 to 2π), we will adjust the window settings to graph only those values. In addition, when the values of these four ratios are listed in tabular form, the maximum and minimum y-values occur at $\tan \theta = \sqrt{3}, -\sqrt{3}$ and $\cot \theta = -\sqrt{3}, \sqrt{3}$ -when $\theta = 60, 120, 240, 300^\circ$ for the $\tan \theta$ and $\theta = 30, 150, 210, 330^\circ$ for the $\cot \theta$.

From this, set the following window settings on your calculator:

- 1) Press **WINDOW**, Xmin = -0.05, Xmax = 6.353, Ymin = -1.8, Ymax = 1.8
- 2) Press **MODE** and select “Radian”
- 3) Press **Y=**, **sin**, x , **GRAPH**
- 4) Press **Y=**, **cos**, x , **GRAPH**
- 5) Press **Y=**, **tan**, x , **GRAPH**

Notice, on some calculators, the calculator inserts vertical lines for the undefined values of the $\tan \theta$ at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

To graph the characteristic curves for the $\csc \theta$, $\sec \theta$, and $\cot \theta$ type the following:

1. **Y=**, **(**, **sin**, x , **)**, **)**, **x⁻¹**

-Recall that the $\sin \theta$ and $\csc \theta$ are reciprocal ratios. Since the calculator does not have buttons for graphing the $\csc \theta$, $\sec \theta$ or $\cot \theta$, we can view their graphs by using **x⁻¹** key, which is the reciprocal of a value (recall laws of exponents).

-Note also that **2nd sin** is not the $\csc \theta$ but the [arcsine](#) feature of the calculator for finding angle measures given a trigonometric value. This is an important distinction in notation for trigonometric curves and equations:

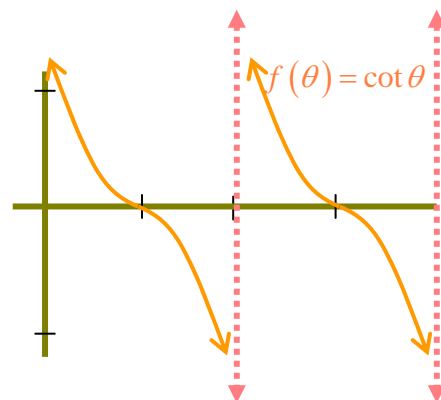
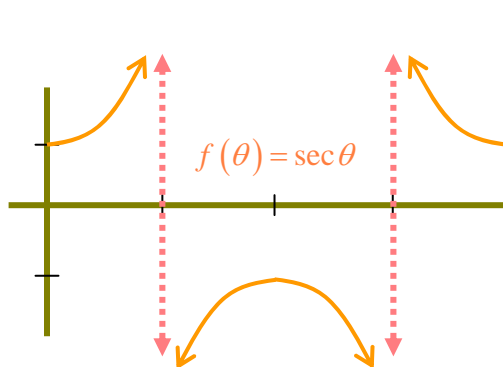
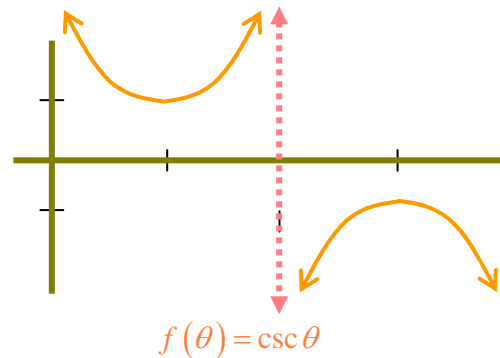
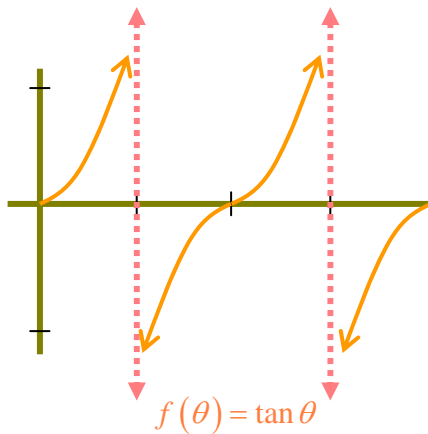
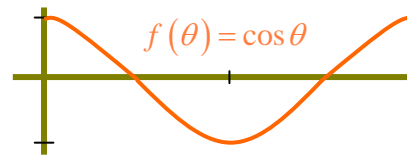
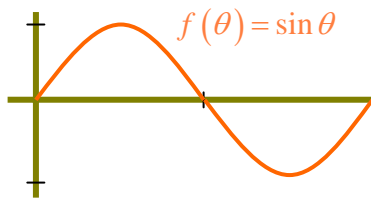
To summarize:

$\sin^{-1} \theta = \text{arcsine of } \theta$: This feature finds an angle measure given the ratio of the opposite leg and hypotenuse in a right triangle (same feature for $\cos \theta$ and $\tan \theta$).

$\sin \theta$: This feature finds the ratio of the opposite leg and hypotenuse given an acute angle measure in a right triangle.

$(\sin \theta)^{-1}$: This evaluates the reciprocal ratio of the $\sin \theta$ which is the $\csc \theta$.

After you view all six trigonometric graphs on the graphing calculator, the following diagrams serve as the characteristic curves for our graphs for all angles from 0 to 2π (0 to 360) and all ratios for all inscribed triangles on the unit circle.



The Standard Form of a Trigonometric Equation (Function)

Now that a characteristic curve is identified for each trigonometric ratio, the next step is to modify each curve through various mathematical operations. In order to view and understand what changes may occur for each curve through mathematics, it is important to establish a standard form for each trigonometric function and classify the various components of this standard equation.

Standard Trigonometric Equations

The standard form for each of the six trigonometric functions are listed as:

$$f(\theta) = A \sin(F\theta + P) \qquad f(\theta) = A \csc(F\theta + P)$$

$$f(\theta) = A \cos(F\theta + P) \qquad f(\theta) = A \sec(F\theta + P)$$

$$f(\theta) = A \tan(F\theta + P) \qquad f(\theta) = A \cot(F\theta + P)$$

For each of the six ratios, the values of **A**, **F** and **P** take on specific tasks and serve to alter the curve's normal appearance in a specific manner. In order to understand the role played by each value, we will examine their significance individually.

Amplitude (A)

The first value examined in a standard equation is the value of “**A**”. **A** is called the “**Amplitude**” of the function. This value determines the maximum and minimum vertical values of the function. In the characteristic curve for the $\sin \theta$, the amplitude is, $A = 1$.

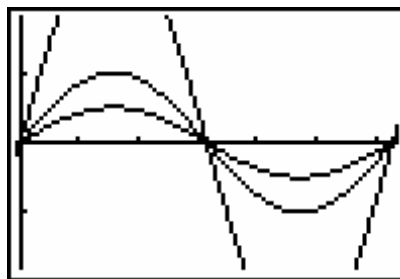
With your calculator's window settings still set for the values used to find the characteristic curves in the first part of this unit, enter the following equation into and view their graphs:

Note: Be sure your calculator is set for “Radian” mode as well.

$$Y_1 = \sin x$$

$$Y_2 = 3 \sin x$$

$$Y_3 = 0.5 \sin x$$



The first curve to be graphed was the characteristic curve for $y = \sin \theta$. The second curve for $y = 3\sin \theta$ is off the screen vertically because, with an amplitude of “3”, it reaches its y – maximum and y – minimum values of 3 and -3 beyond the calculator’s window settings for Ymin and Ymax. (To see this entire curve, you may want to adjust these settings accordingly). The third curve graph reaches a maximum and minimum value of $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. An important feature for all curves is that they all reach their maximum and minimum values at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ and all three cross the x -axis at $0, \pi$ and 2π . A change in a trigonometric curve’s amplitude “stretches” the curve vertically, but does not affect θ values where it will obtain maximum, minimum, and zero values.

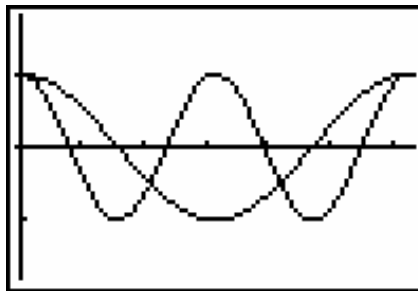
Frequency and Period (F and F^{-1})

The next component of the standard equation is the value of “F” the “Frequency” and the reciprocal, “ F^{-1} ”, which is called the “Period”.

In their characteristic graphs, each trigonometric ratio is said to exhibit **one** complete “Cycle” or “Period” from 0 to 2π with a “Frequency” of **one**. Both of these values can be adjusted by altering the value of “F” in the standard equation. Type the following equations into your calculator and view each graph.

$$Y_1 = \cos x$$

$$Y_2 = \cos 2x$$



When Y_2 is graphed, two complete cosine curves appear between 0 and 2π . If you type a third equation into Y_3 such as: $Y_3 = \cos(0.5x)$ and view its graph, only the first half of the cosine’s characteristic curve will appear. Therefore, the value of “F”-the frequency- determines how many characteristic curves will appear in a standard viewing window from 0 to 2π .

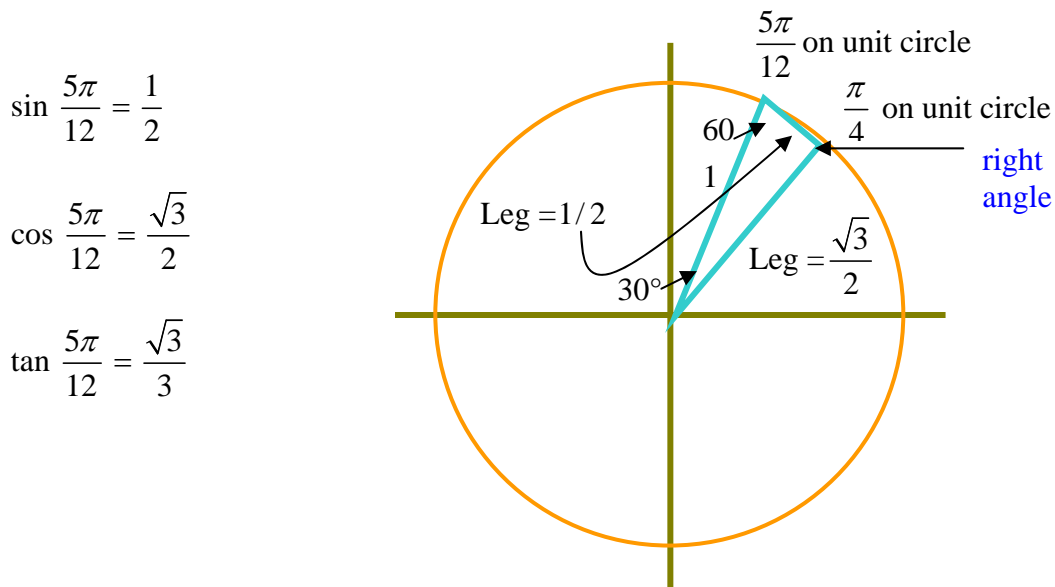
Related to the frequency is the **Period**. As stated above, the Period is the reciprocal of the Frequency (F^{-1}).

In the example above, $F^{-1} = \frac{1}{2}$ for $Y_2 = \cos(2x)$ and $F^{-1} = 2$ for $Y_3 = \cos(0.5x)$. The period determines how much of the x -axis must be used in order to view one

characteristic curve. For $Y_2 = \cos 2x$ above, two complete cosine curves are seen between 0 and 2π . This indicates that one complete cosine curve is graphed between 0 and π , which is only *half* the usual horizontal distance. Since one graph is completed in *half* the normal values, the period for $y = \cos 2x$ is $F^{-1} = \frac{1}{2}$ for $Y_3 = \cos(0.5x)$ only *half* of the cosine's characteristic curve is seen. In order to view the entire curve, we would need to set the x -maximum at double its current setting or $X_{\max} = 4\pi$ (this can be done on your own). Since it requires twice the number of x -values to view $Y_3 = \cos(0.5x)$ as one complete curve, we conclude that the Period for this equation is: $F^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$.

Phase Shift (P)

For all six trigonometric ratios, the first critical value is $\theta = 0$. At $\theta = 0$ the radius of the unit circle is laying on the x -axis between quadrants I and IV. Had we so desired, however, these values could have started at any angle measure on the unit circle. In the example below, the values of the six ratios begin at $\frac{\pi}{4} = 45^\circ$. From here, θ can increase by $\frac{\pi}{6} = 30^\circ$ to $75^\circ = \frac{5\pi}{12}$ and a 30-60-90 right triangle inscribed at this value is the starting point for evaluating the six trigonometric ratios for $\theta = \frac{\pi}{4}$.



The values for the remaining values could then be calculated as before throughout one rotation of the unit circle. From this, the new starting point, each characteristic curve would now start at $\frac{\pi}{4}$ and the new ending point would be $2\pi + \frac{\pi}{4} = \frac{9\pi}{4} = 405^\circ$.

Changing the starting and ending point of each trigonometric ratio is to cause a “Phase Shift (P)” to the standard equation. Since a Phase Shift causes the starting and ending values on the unit circle to change, the value of P is usually recorded or entered as an angle in radian measure in the standard equation.

To view the above example on your calculator for the $\sin \theta$, type the following and graph:

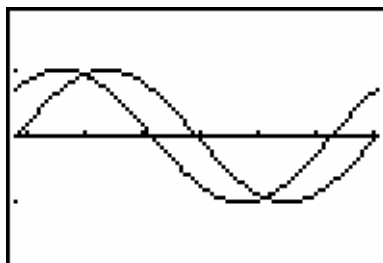
$$Y_1 = \sin x$$

$$Y_2 = \sin \left(x - \frac{\pi}{4} \right)$$

For the $\sin \left(x - \frac{\pi}{4} \right)$ to begin graphing the function as though $\frac{\pi}{4}$ were the beginning value we subtract this value inside the parenthesis. In this way when $x = \frac{\pi}{4}$ in $Y_2 = \sin \left(x - \frac{\pi}{4} \right)$ we obtain, $\sin \left(\frac{\pi}{4} - \frac{\pi}{4} \right) = \sin 0 = 0$ as though $\frac{\pi}{4}$ were equivalent to 0 in the characteristic curve. To see all of $Y_2 = \sin \left(x - \frac{\pi}{4} \right)$ phase shifted, reset your window settings to the following values:

$$X_{\min} = \frac{\pi}{4} - 0.01 \qquad X_{\max} = \frac{9\pi}{4} + 0.01$$

Note: Be sure your calculator is set for “Radian” mode as well.



A Phase Shift can occur either before or after $\theta = 0^r$. If we begin calculating our six trigonometric ratios before $\theta = 0^r$ such as $\theta = 0 - \frac{\pi}{4} = 45^\circ$ then continue around the unit circle, the Phase Shift is said to be “negative”, when the calculations begin after $\theta = 0^r$ the Phase Shift is “positive”. In the above example $x - \frac{\pi}{4}$ indicated a positive phase shift of $P = \frac{\pi}{4}$. To view a negative phase shift type the following into Y_3 :

$$Y_3 = \sin \left(x + \frac{\pi}{4} \right)$$

Note: For all trigonometric functions in standard form the value “ $Fx + P$ ” is called the “argument” of the angle. From this discussion we conclude the following about the Phase Shift of a trigonometric function:

- 1.) A Phase Shift changes the starting and ending values of the characteristic curve.
- 2.) If the argument of a trigonometric curve is of the form “ $Fx + P$ ” then the Phase Shift is negative.
- 3.) If the argument of a trigonometric curve is of the form “ $Fx - P$ ” then the Phase Shift is positive.

Graphing Trigonometric Functions for all Values of A, F, and P in the Standard Equation

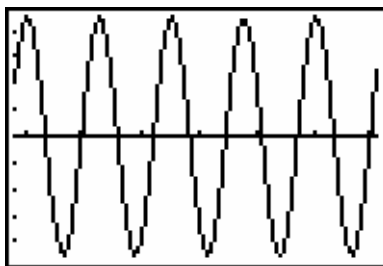
In the previous discussion, we examined the individual roles played by A, F and P in the Standard Equation of a trigonometric curve. In most cases, these values do not occur individually. Consider the following equation:

$$f(x) = (9/2)\cos(5x + (5\pi/12))$$

To view this curve on your calculator with the window settings used to view the characteristic curve for the cosine enter the following:

$$Y_1 = (9/2)\cos(5x + (5\pi/12)); \quad Y_{\min} = -4.52 \quad Y_{\max} = 4.52$$

$$\text{Recall from a previous exercise:} \quad X_{\min} = \frac{\pi}{4} - 0.01 \quad X_{\max} = \frac{9\pi}{4} + 0.01$$



Although this equation graphs the cosine function, the displayed graph does not begin at the maximum value of the cosine as does the characteristic curve. In order to analyze this graph correctly, all values of the equation must be identified and interpreted.

Example #1: Identify the Amplitude, Frequency, Period, Phase Shift, Starting Point and Ending Points of

$$y = \frac{9}{2}\cos\left(5\theta + \frac{5\pi}{12}\right).$$

Select an appropriate viewing window and graph one characteristic curve for this equation.

Step #1: Identify Amplitude $A = \frac{9}{2} = 4.5$

Identify Frequency $F = 5$

Identify Period $F^{-1} = \frac{1}{5}$

Identify Phase Shift $P = -\frac{5\pi}{12}$

Step #2: Calculate starting and ending points for the graph.

A.) The starting point is equivalent to the argument, $5x + \frac{5\pi}{12} = 0$

Solving for x we obtain:

$$5x + \frac{5\pi}{12} = 0$$

$$5x = -\frac{5\pi}{12}$$

$$x = -\frac{\pi}{12} = -15^\circ$$

B.) The ending point is equivalent to calculating all 17 critical values of the function through one rotation of the unit circle beginning at

$-\frac{\pi}{12} = -15^\circ$. This occurs when the argument is set equal to 2π or:

$$5x + \frac{5\pi}{12} = 2\pi$$

$$5x = \frac{19\pi}{12}$$

$$x = \frac{19\pi}{60} = 57^\circ$$

Step #3: Graph one complete curve for the function by setting the calculator to the appropriate viewing window based on the values found in Steps #1 and #2 above.

A.) Amplitude = $\frac{9}{2} = 4.5$. Since this represents a vertical shift in the maximum and minimum values of $\cos \theta$, we enter the following:

$$Y_{\min} = -4.5 - 0.05$$

$$Y_{\max} = 4.5 + 0.05$$

B.) The calculated starting and ending values of the function are:

$$\text{-starting value} = -\frac{\pi}{12} = -15^\circ$$

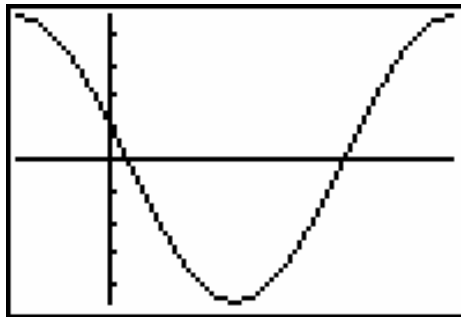
$$\text{-ending value} = \frac{19\pi}{60} = 57^\circ$$

*Since angle values are found on the x -axis, we enter the following:

$$X_{\min} = -\frac{\pi}{12} - 0.05$$

$$X_{\max} = \frac{19\pi}{60} + 0.05$$

C.) The resulting graph of one complete curve is shown below.



Example #2: Calculate the Amplitude, Frequency, Period, Phase Shift, Starting and Ending values of:

$$f(\theta) = \frac{4}{5} \tan\left(\frac{7}{3}\theta - \frac{\pi}{6}\right)$$

Step #1: $A = \frac{4}{5}$: $F = \frac{7}{3}$: $F^{-1} = \frac{3}{7}$: $P = \frac{\pi}{6}$

Step #2:

A.) Starting value: $\frac{7}{3}x - \frac{\pi}{6} = 0$

$$x = \frac{\pi}{14} \approx 12.857^\circ$$

B.) Ending value: $\frac{7}{3}x - \frac{\pi}{6} = 2\pi$

$$\frac{7}{3}x = 2\pi + \frac{\pi}{6}$$

$$\frac{7}{3}x = \frac{13\pi}{6}$$

$$x = \frac{13\pi}{14} \approx 167^\circ$$

Step #3:

A.) $Y_1 = (4/5) \tan((7/3)x - \pi/6)$

B.) $X_{\min} = \frac{\pi}{14} - 0.05$ $X_{\max} = \frac{13\pi}{14} + 0.05$

C.) $Y_{\min} = \frac{-4}{5} - 0.05$ $Y_{\max} = \frac{4}{5} + 0.05$

