SPECIAL RIGHT TRIANGLES AND CIRCULAR TRIGONOMETRY

In a previous unit we learned that every acute angle measure in a right triangle is uniquely associated with the reduced ratio of the sides of that triangle. For all the possible measures for these acute angles greater than zero degrees and less than ninety degrees, three angle measures are deemed to be critical because of the special ratios that exist in the sides of triangles when these angles are present. This unit will examine these "critical values" in the two special right triangles where they are found. To further examine these special ratios, the right triangles will be inscribed at various positions inside a circle which is centered at the origin of the *xy*-plane. From this diagram, values for the six trigonometric ratios for these triangles will be deduced and later graphed using the graphing calculator. Once graphed, the standard curves of the six trigonometric ratios can be analyzed.

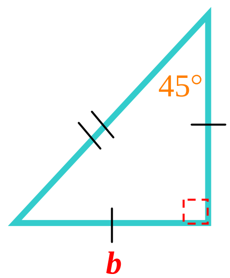
Special Right Triangles

Isosceles Right Triangle (45-45-90)

From previous math courses, you may recall that an isosceles triangle is a triangle with two equal sides and two equal angles. If the two legs of a right triangle are congruent, then the right triangle is also isosceles and each acute angle measure is equal to $45° \Rightarrow \frac{\pi^r}{4}$.

45 – 45 – 90

Right Triangle



In this triangle, the ratio of the legs is $\frac{a}{b} = \frac{b}{a} = \frac{a}{a} = \frac{b}{b} = 1$

From this we conclude that the $\tan 45° = \cot 45° = 1$. Therefore, $45° \left(\frac{\pi^r}{4}\right)$ is the first critical value of the acute angles between $0° \left(0^r\right)$ and $90° \left(\frac{\pi^r}{2}\right)$.

To find the ratio of the legs and the hypotenuse in this triangle, we solve for "c" using the Pythagorean Theorem: $a^2 + b^2 = c^2$

Let $b = a$, then

$$a^2 + a^2 = c^2$$
$$2a^2 = c^2$$
$$a\sqrt{2} = c$$

Dividing by "a" gives the following ratios:

$$\sqrt{2} = \frac{c}{a} \text{ and } \frac{a}{c} = \frac{\sqrt{2}}{2} \textbf{ note: } \left(\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\right)$$

Since $b = a$ in this triangle, the sin 45 = cos 45 = $\frac{a}{c} = \frac{b}{c} = \frac{\sqrt{2}}{2}$

and csc 45 = sec 45 = $\frac{c}{a} = \frac{c}{b} = \sqrt{2}$

Scalene Right Triangle (30-60-90)

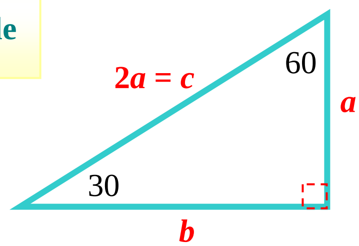
The second special right triangle occurs when the ratio of the angles in the triangle are $1:2:3$ (30-60-90). When the angles of the triangle are divided in this fashion, the ratios of the lengths of the sides are $1: \ 2: \ \sqrt{3}$. This can also be shown using the Pythagorean Theorem. The actual derivation of the angles $30° \left(\frac{\pi}{6}\right)$ and $60° \left(\frac{\pi}{3}\right)$ can be found using the $\sin^{-1}\theta$ & $\cos^{-1}\theta$. The lengths of the sides can be found in the following manner:

Construct a right triangle so that the length of the hypotenuse is twice the length of one leg of the triangle $(2a = c)$. Use the Pythagorean Theorem to find an expression for side "b".

30 – 60 – 90

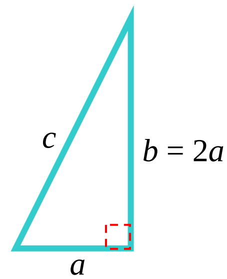
Right Triangle

$$
\begin{aligned}
a^2 + b^2 &= (2a)^2 \\
a^2 + b^2 &= 4a^2 \\
b^2 &= 3a^2 \\
b &= a\sqrt{3}
\end{aligned}
$$



Although the angles are labeled in the above diagram in their correct positions, the actual determination of these angles' values is not readily apparent simply by finding the ratios of the sides. 'If a right triangle were constructed by allowing the legs to have a ratio of 2*a* = *b*, then the Pythagorean Theorem would yield:

$$(2a)^2 + a^2 = c^2$$
$$4a^2 + a^2 = c^2$$
$$a\sqrt{5} = c$$

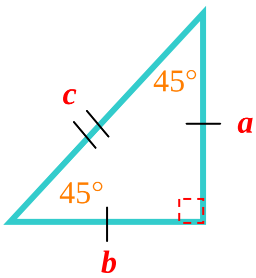


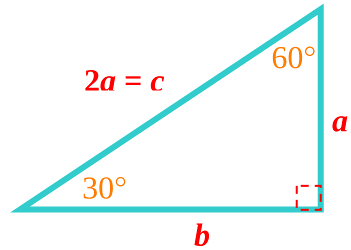
In this situation the two acute angles would measure approximately $63.435°$ and $26.565°$. Although these values can be obtained using the arcsine, arccosine or arctangent functions on your calculator, the actual mathematical determination of these values is outside the scope of this course. Because trigonometry emphasizes angle measures that are uniquely associated with ratios, the 30-60-90 right triangle becomes special to our study because of the whole number ratios of the angles.

Determining the $\sin 60$, $\cos 30$, $\tan 30$, $\csc 60$, and etc. will be emphasized later in the unit when we associate each special right triangle with circular trigonometry. Before proceeding we will practice using the ratios found in those triangles to find the lengths of unknown sides in right triangles.

Lengths of Sides of Special Right Triangles

In the first part of this unit the ratios of the sides of the two special right triangles were derived. These ratios are summarized below:



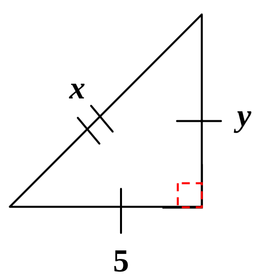


$$\boxed{\begin{aligned} a &= b \\ a\sqrt{2} &= c \end{aligned}}$$

$$\boxed{\begin{aligned} 2a &= c \\ a\sqrt{3} &= b \end{aligned}}$$

The ratios in each boxed area above can be used as *formulas* to find the lengths of unknown sides in each special right triangle.

Example #1: Find the unknown lengths in the triangle below.



Although the angle measures are not labeled the equal hatch marks in the legs indicate this to be an isosceles right triangle. Using the formulas for a 45-45-90 right triangle, the unknown lengths are:

1) $a = b = 5 = y \Rightarrow y = 5$
2) $a\sqrt{2} = c = x \Rightarrow x = 5\sqrt{2}$

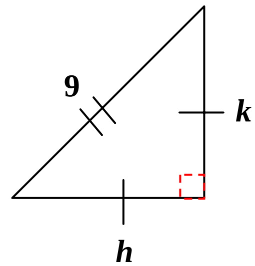
Example #2: Find *h* and *k*.

$$a = b = h = k$$

$$c = 9$$

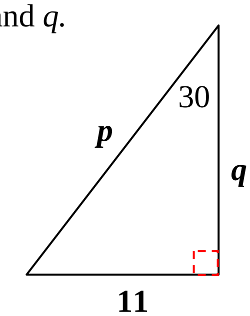
$$\Rightarrow a\sqrt{2} = 9$$

$$a = \frac{9}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{9\sqrt{2}}{2}$$



$$\Rightarrow h = k = \frac{9\sqrt{2}}{2}$$

Example #3: Find *p* and *q*.



Note: In a 30-60-90 right triangle, "*a*" will always be opposite the $30°$ angle and "*b*" will always be opposite the $60°$ angle.

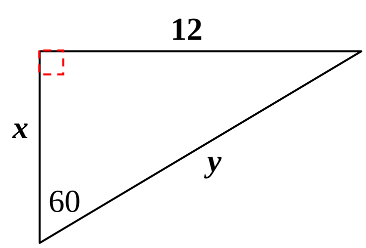
Therefore $a = 11;\ c = p;\ b = q$

1) $2a = c$
$2(11) = c$
$22 = c = p$

2) $a\sqrt{3} = b$
$11\sqrt{3} = b = q$

Example #4: Find *x* and *y*.

$$a = x$$
$$b = 12$$
$$c = y$$



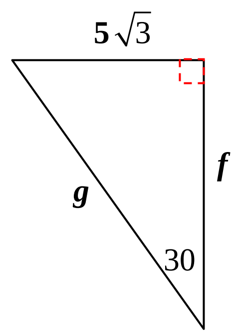
Therefore: $x\sqrt{3} = 12$

$$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

and

$$2x = y \Rightarrow 2(4\sqrt{3}) = 8\sqrt{3} = y$$

Example #5: Find g and f.



$a = 5\sqrt{3}$
$b = f$
$c = g$

1) $2 \times (5\sqrt{3}) = g$

$10\sqrt{3} = g$

2) $a\sqrt{3} = f$

$(5\sqrt{3})\sqrt{3} = f$

$15 = f$

Circular Trigonometry

The previous discussions have established the following angle measures as critical values in the study of trigonometry:

$$0^\circ = 0^r$$

$$30^\circ = \frac{\pi}{6}^r$$

$$45^\circ = \frac{\pi}{4}^r$$

$$60^\circ = \frac{\pi}{3}^r$$

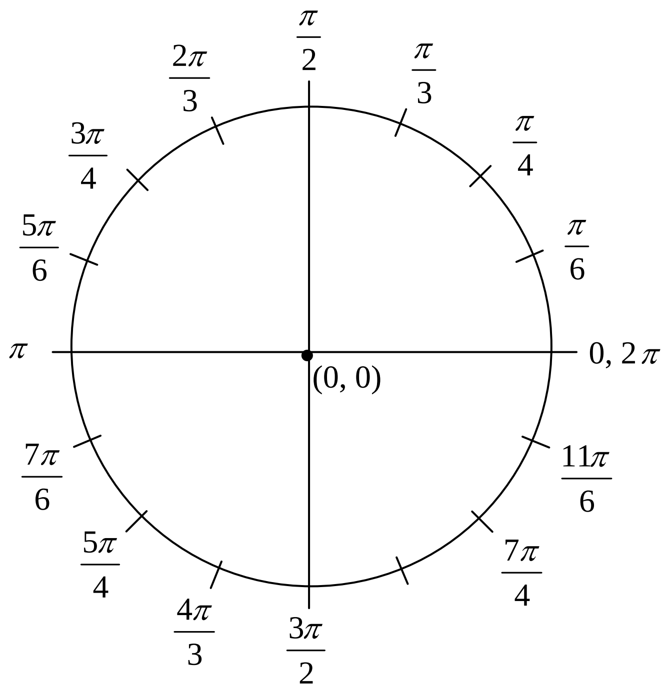
$$90^\circ = \frac{\pi}{2}^r$$

In addition all whole number multiples will also be critical values for all angles $\leq 360^\circ$.

Example #1:

$$7(30^\circ) = 210^\circ \;\Rightarrow\; 7\left(\frac{\pi}{6}\right) = \frac{7\pi}{6} \qquad \textit{and} \qquad 5(45^\circ) = 225^\circ \;\Rightarrow\; 5\left(\frac{\pi}{4}\right) = \frac{5\pi}{4}$$

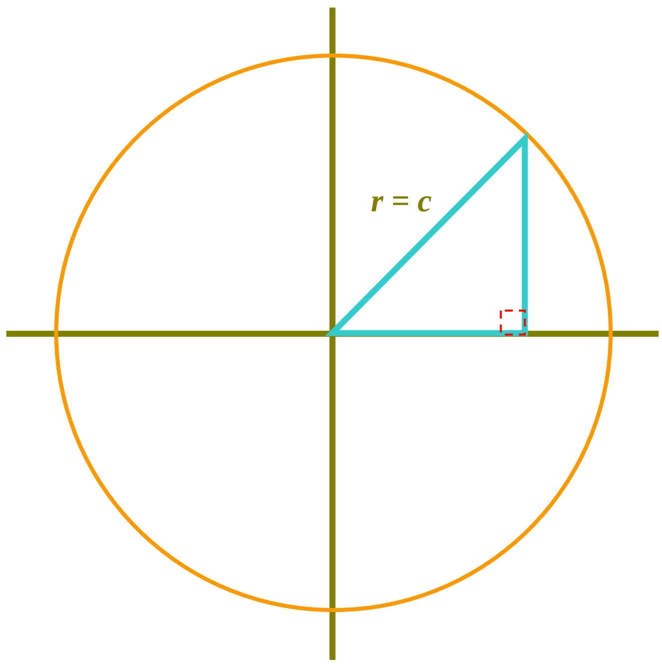
Recall that $2\pi^r = 360°$ is the radian-degree measure of a circle. Successive multiplication of the critical values will provide all critical value angle measures in a circle. Also recall that successive multiplications by the imaginary number "i" caused a counterclockwise rotation of an imaginary number, (vector), through a circle. Since imaginary and complex numbers are especially tied to trigonometry (which will be studied in a later unit), the whole number multiples of the critical values less than or equal to $360°$ $(2\pi^r)$ will be labeled counterclockwise on a circle as well. In addition, the circle will be drawn on the *xy*-plane with the origin (0, 0) as the center of the circle. The diagram with critical values labeled in radian angles is shown below:



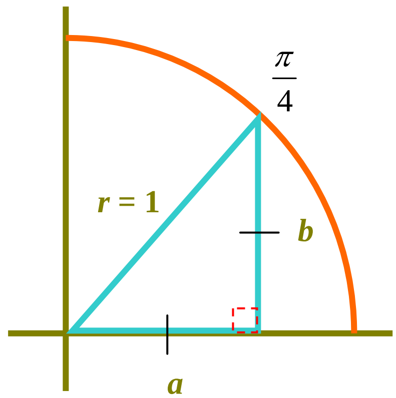
The Unit Circle

The key value of any circle is its radius. In mathematics, a circle with a radius equal to one $(r = 1)$ is called a "Unit Circle".

As established in a previous unit, the reduced ratios of a triangle are uniquely associated with an acute angle in a right triangle. In the circle diagram below, a right triangle can be inscribed in the circle in the following manner:



In the above diagram, the hypotenuse of the triangle is also the radius of the circle. If we allow r = 1 and select one of our two special right triangles to be inscribed in each quadrant of the circle, the six trigonometric values can be established for each critical value from 0^r to $2\pi^r$. To see how this is accomplished, we begin by inscribing and isosceles right triangle in the circle in quadrant #1.



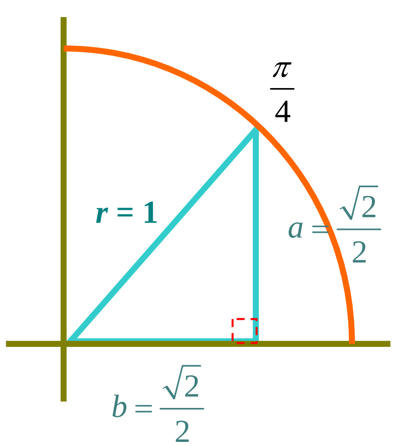
Notice that since $\frac{\pi}{4} = 45°,$ the radius of the circle, (hypotenuse of the right triangle) intersects the circle at this point.

Using the formulas for a 45-45-90 triangle, values for "*a*" and "*b*" can be determined:

$$a\sqrt{2} = c \Rightarrow a\sqrt{2} = 1$$

$$a = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = b$$

The inscribed 45-45-90 triangle is now completely labeled as:



We now use the origin as the vertex of our reference angle to find the values of our six trigonometric ratios.

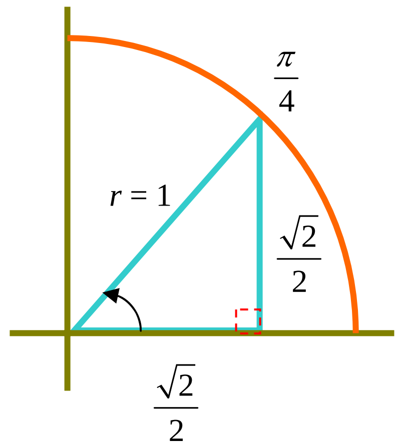
Example #2: Find the six trigonometric ratios for $\frac{\pi}{4}$ using the unit circle and associated inscribed 45-45-90 right triangle

Step #1: Identify: hypotenuse, opposite leg, and adjacent leg for the reference angle

hypotenuse = 1

opposite leg = $\frac{\sqrt{2}}{2}$

adjacent leg = $\frac{\sqrt{2}}{2}$



Step #2: Use the definitions of the six trigonometric ratios.

$$\sin\frac{\pi}{4} = \frac{opp}{hyp} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

$$\csc\frac{\pi}{4} = \frac{hyp}{opp} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\cos\frac{\pi}{4} = \frac{adj}{hyp} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$$

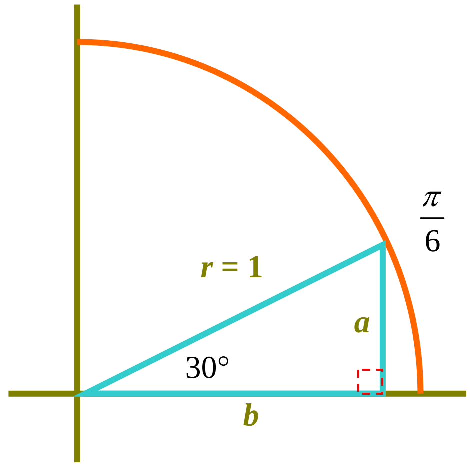
$$\sec\frac{\pi}{4} = \frac{hyp}{adj} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = \frac{opp}{adj} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$\cot\frac{\pi}{4} = \frac{adj}{opp} = 1$$

Example #3: Find six trigonometric ratios for $\frac{\pi}{6}$

For this angle we inscribe a 30-60-90 right triangle in the circle with the vertex of the $30°\left(\frac{\pi}{6}\right)$ angle at the origin.

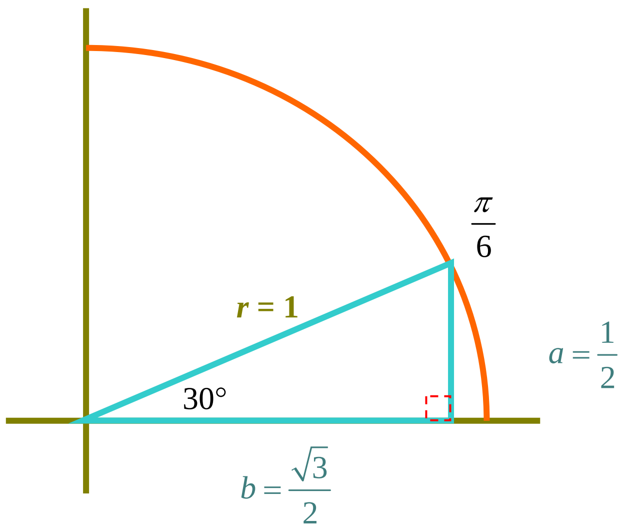


Using the formulas for the sides of a 30-60-90 right triangle, the following values are established for "a" and "b"

1) $$2a = c$$
$$2a = 1$$
$$a = \frac{1}{2}$$

2) $$a\sqrt{3} = b$$
$$\frac{1}{2}\sqrt{3} = b$$
$$\frac{\sqrt{3}}{2} = b$$

The fully labeled inscribed triangle for $\frac{\pi}{6}$ is now:



The six trigonometric ratios for $\frac{\pi}{6}$ are:

$$\sin\frac{\pi}{6} = \frac{\frac{1}{2}}{1} = \frac{1}{2} \qquad \csc\frac{\pi}{6} = \frac{1}{\frac{1}{2}} = 2$$

$$\cos\frac{\pi}{6} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

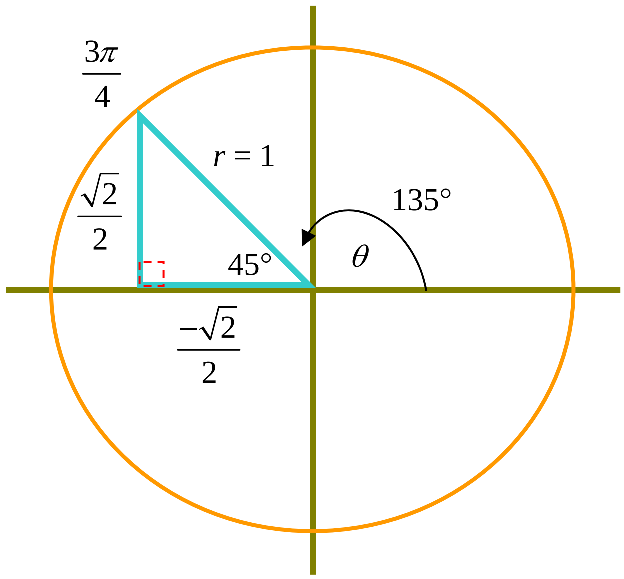
$$\tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{6} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \cdot \frac{2}{1} = \sqrt{3}$$

The trigonometric values for $60°\left(\frac{\pi}{3}\right)$ is left for you to complete in the assignment for this unit.

Right Triangles and Reference Angles in Quadrants #2,3,4

As the special right triangles are inscribed in quadrants #2, 3, and 4, the legs of the triangles take on values to reflect the direction taken in order for the radius (hypotenuse) to intersect the critical value on the unit circle. In addition, the measures of the angles in those quadrants increase beyond $90°\left(\frac{\pi}{2}\right)$.

Consider the following triangle for $\frac{3\pi}{4}$ $(135°)$:



Notice that values of the legs of the inscribed triangle are still $\frac{\sqrt{2}}{2}$. However, the horizontal leg is labeled as negative. This reflects the fact that the triangle is inscribed in Quadrant #2. We also note that the hypotenuse of this triangle (radius of the circle) intersects the critical value of $\frac{3\pi}{4}$ which is $135°$. Therefore, in order to find the values of the six trigonometric ratios for $\frac{3\pi}{4}$, we will use the inscribed triangle's acute $45°\left(\frac{\pi}{4}\right)$ angle -with vertex at the center- as a reference angle. The lengths of the sides of this triangle -with the horizontal side labeled as negative- is used to find the six trigonometric ratios.

Example #1: Find the six trigonometric ratios for $\frac{3\pi}{4}$

$$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$$

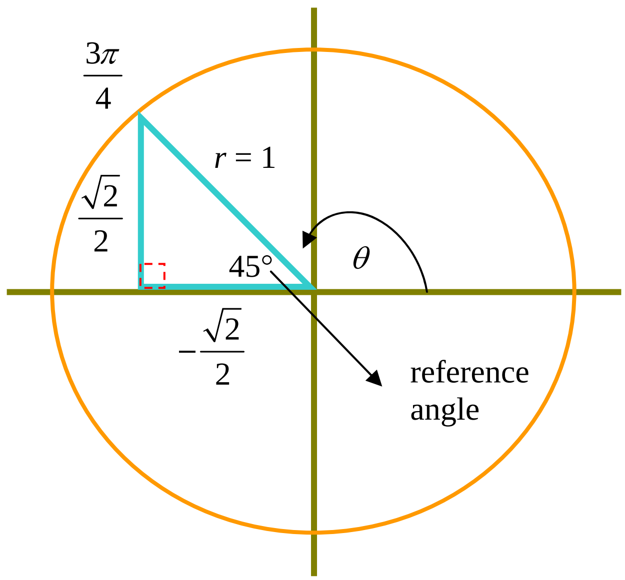
$$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\tan\frac{3\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

$$\csc\frac{3\pi}{4} = \sqrt{2}$$

$$\sec\frac{3\pi}{4} = -\sqrt{2}$$

$$\cot\frac{3\pi}{4} = -1$$



Example #2: Find the six trigonometric ratios for $\frac{4\pi}{3}$ in Quadrant #3.

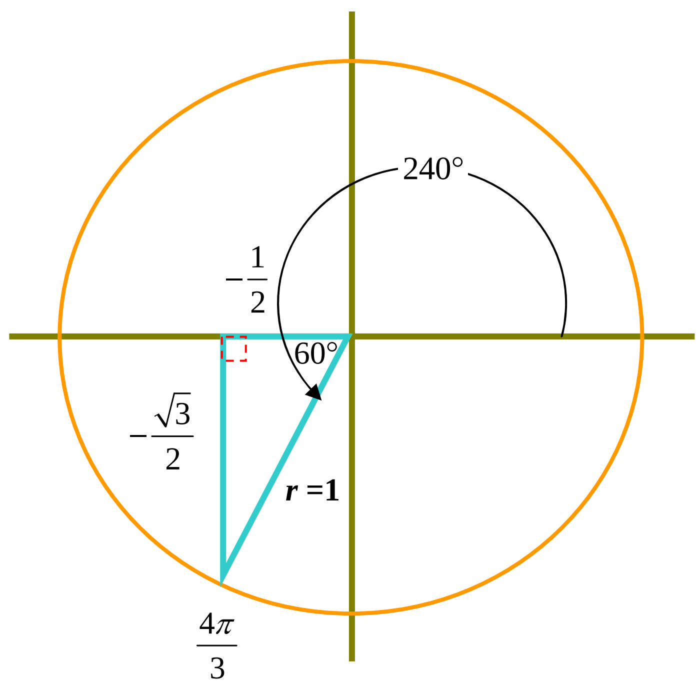
Note that $240^\circ = 180^\circ + 60^\circ$. Since $180^\circ = (\pi^r)$ is the boundary angle between Quadrant #2 and #3, the inscribed triangle for $\frac{4\pi}{3}$ is 30-60-90 with the 60^0 used as our reference angle. The values of the legs of this triangle are know to = $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. This time, both these values are labeled as negative to reflect the triangle's position in Quadrant #3. Also note that the hypotenuse (radius) always equals positive one $(r = 1)$ as it rotates through the circle. As mentioned earlier, trigonometry emphasizes angle measures and relations. As the radius moves counterclockwise through the circle, it will intersect every possible angle between 0 and 2π. It is for this reason that its value always equals positive one but the values of the legs of the inscribed triangle are labeled either positive or negative to indicate the position of the radius in the circle.

From the diagram below and the definitions of the six trigonometric ratios, the values for $\frac{4\pi}{3}$ are as follows:

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ $\qquad$ $\csc\frac{4\pi}{3} = -\frac{2\sqrt{3}}{3}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$ $\qquad$ $\sec\frac{4\pi}{3} = -2$

$\tan\frac{4\pi}{3} = \sqrt{3}$ $\qquad$ $\cot\frac{4\pi}{3} = \frac{\sqrt{3}}{3}$



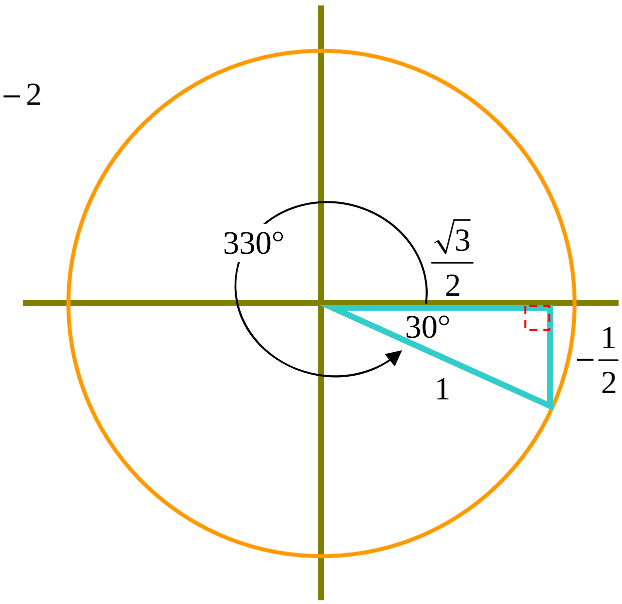
Example #3: Find the six trigonometric ratios for $\frac{11\pi}{6}\ (330°)$ in Quadrant #4.

Continuing with the same reasoning in the previous examples, the following inscribed triangle and ratios are obtained.

$$\sin\frac{11\pi}{6} = -\frac{1}{2} \qquad \csc\frac{11\pi}{6} = -2$$

$$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sec\frac{11\pi}{6} = \frac{2\sqrt{3}}{3}$$

$$\tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3} \qquad \cot\frac{11\pi}{6} = -\sqrt{3}$$



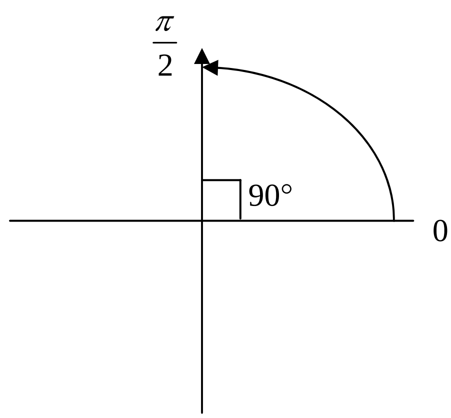
The remaining ratios for all other critical values in the unit circle are left to do in the assignment for this unit.

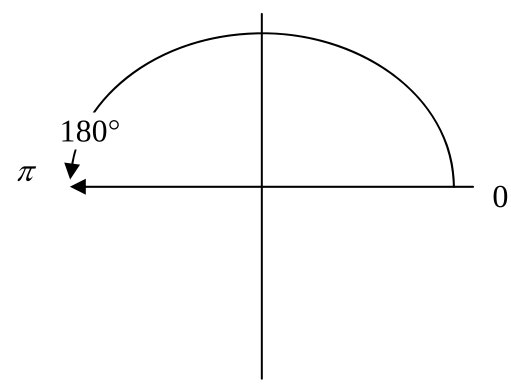
Special Trigonometric Ratios

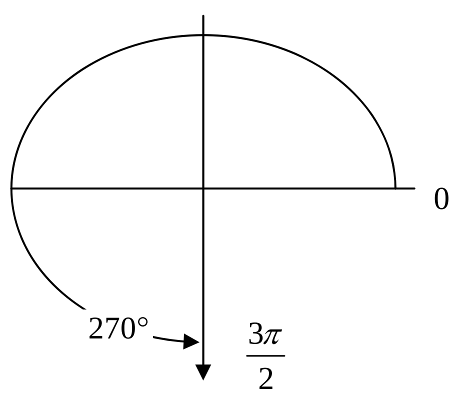
Trigonometric Ratios for $0,\ \pi/2,\ \pi,\ 3\pi/2,\ 2\pi$

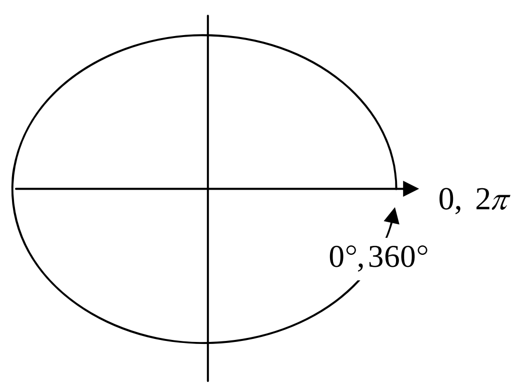
In the previous discussion, inscribed triangles were used to evaluate the trigonometric ratios of the angles found in each quadrant of the unit circle. However, the angle measures of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$ and 2π $(0°, 90°, 180°, 270°, 360°)$ are not found “inside” any of the Quadrants # 1, 2, 3, or 4. Each of those angle’s position on the unit circle is found on either the *x* or *y*-axis which are the boundary lines between the four quadrants.

Consider the following diagrams:





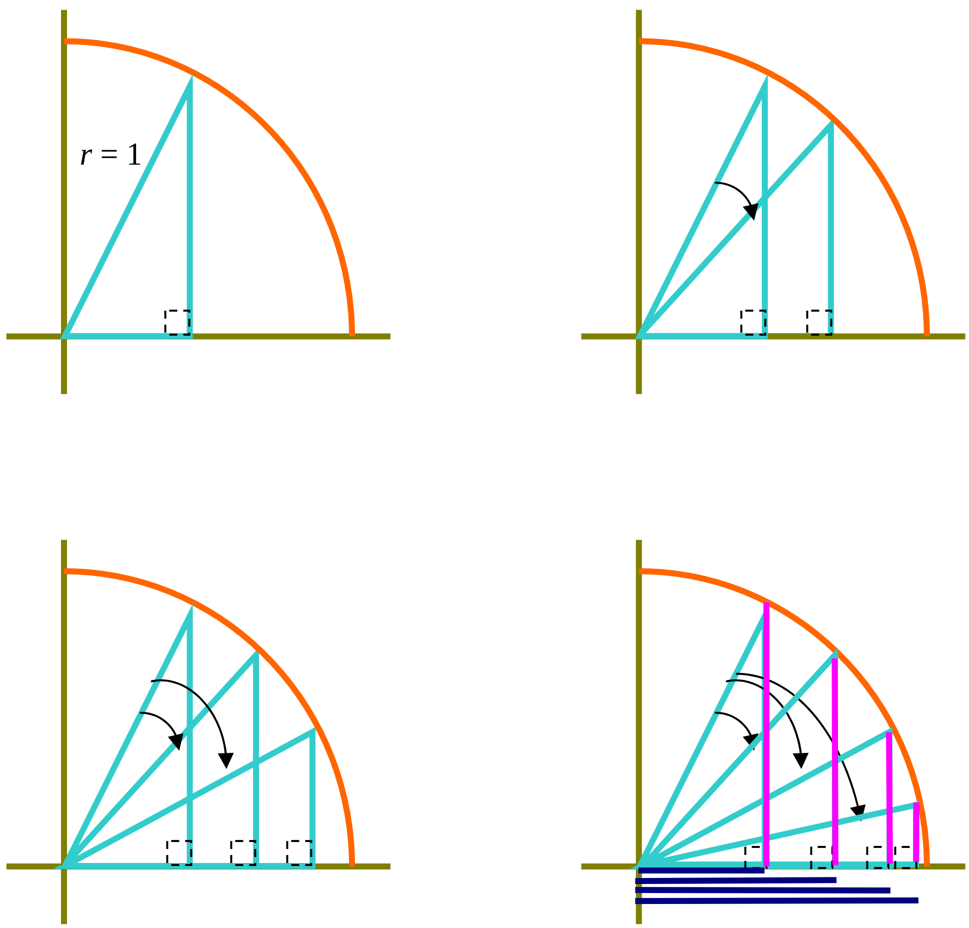




For each of these diagrams, the radius (which would be the hypotenuse of an inscribed right triangle) lies on either the *x* or *y*-axis. When the radius is on one of the axes, it is not possible to inscribe a right triangle in the circle and use a reference angle centered at $(0,0)$ to evaluate the trigonometric ratios.

To find these trig values, we will borrow the concept of the "limit" from Calculus.

Consider the following sequence of diagrams for inscribed triangles in Quadrant #1 where the radius (hypotenuse) recedes to the *x*-axis.



Notice that as the radius recedes closer to the *x*-axis, the lengths of the legs of the right triangles change in length. Specifically the **opposite leg** *reduces its length* and the **adjacent leg** *increases in length.*

When the radius comes to rest on the *x*-axis, the length of each of the triangles eventually takes on the following values:

Hypotenuse (radius)	= 1
Opposite leg	= 0
Adjacent leg	= 1

These final values are called the "limits" of the process for establishing the lengths of the sides of a right triangle needed to evaluate the six trigonometric ratios. When the radius finally comes to rest on the *x*-axis, the value of the reference angle in the triangles has also approached a limit of $0°$. Using these "limiting values", we can evaluate our six trigonometric ratios in the following way:

$$\sin 0° = \frac{opp\ \ leg}{hypotenuse} = \frac{0}{1} = 1 \qquad \csc 0° = \frac{hyp}{opp} = \frac{1}{0} = \text{ undefined}$$

$$\cos 0° = \frac{adj}{hyp} = \frac{1}{1} = 1 \qquad \sec 0° = \frac{hyp}{adj} = \frac{1}{1} = 1$$

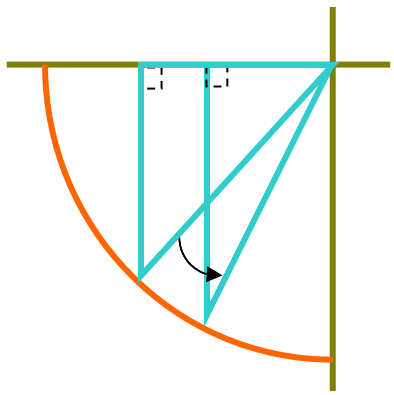
$$\tan 0° = \frac{opp}{adj} = \frac{0}{1} = 1 \qquad \cot 0° = \frac{adj}{opp} = \frac{1}{0} = \text{undefined}$$

Notice that not all six trigonometric ratios can always be evaluated. The $\csc 0°$ and $\cot 0°$ both result in division by 0 which is mathematically undefined. This fact will become important in a later unit when we graph each trigonometric ratio individually.

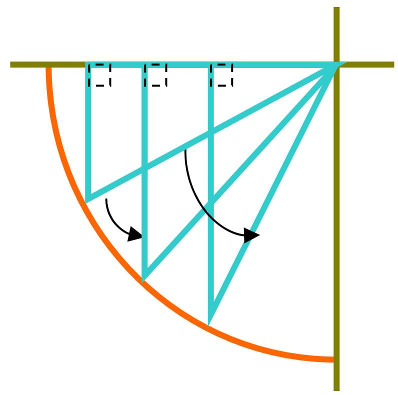
Example #1: Find six trigonometric ratios for $\frac{3\pi}{2}$ $(270°)$ by assessing the limits of the sides of inscribed triangles at $270°$.

Step #1: Inscribe a sequence of right triangles in either quadrant #3 or #4 (for this example a triangle will be inscribed in Quadrant #3. As you follow the example consider how the same final results could be obtained by inscribing the sequence of triangles in Quadrant #4. This may prove useful in the assignment for this unit where you will use this method to evaluate the six ratios for $\frac{\pi}{2}, \pi,$ *and* 2π).

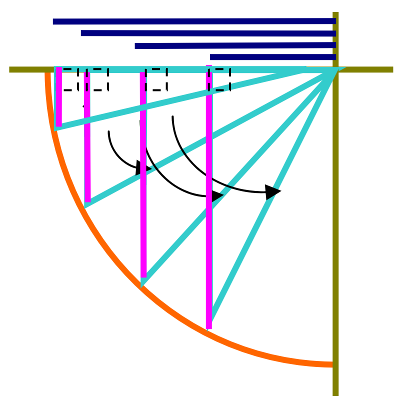
1)



2)



3)



From this sequence we notice that the **opposite leg** is *increasing in length* while the **adjacent leg** *decreases*. Recalling that the legs not only reflect length but direction that locates the radius's position, we obtain the following limiting values:

Step #2: Identify the limiting values for the sides of the inscribed triangle:

a) Hypotenuse = 1
b) Adjacent leg = 0
c) Opposite leg = -1   (why?)

Step #3: Use your results from Step #2 to evaluate the six trigonometric ratios for $\frac{3\pi}{2}$

$\sin\frac{3\pi}{2} = \frac{opp}{hyp} = \frac{-1}{1} = -1$ $\quad\quad$ $\csc\frac{3\pi}{2} = \frac{hyp}{opp} = \frac{1}{-1} = -1$

$\cos\frac{3\pi}{2} = \frac{adj}{hyp} = \frac{0}{1} = 0$ $\quad\quad$ $\sec\frac{3\pi}{2} = \frac{hyp}{adj} = \frac{1}{0} = \text{undefined}$

$\tan\frac{3\pi}{2} = \frac{opp}{adj} = \frac{-1}{0} = \text{ undefined}$ $\quad\quad$ $\cot\frac{3\pi}{2} = \frac{adj}{opp} = \frac{0}{-1} = 0$

Evaluation of the remaining angles is left as an exercise for this unit.