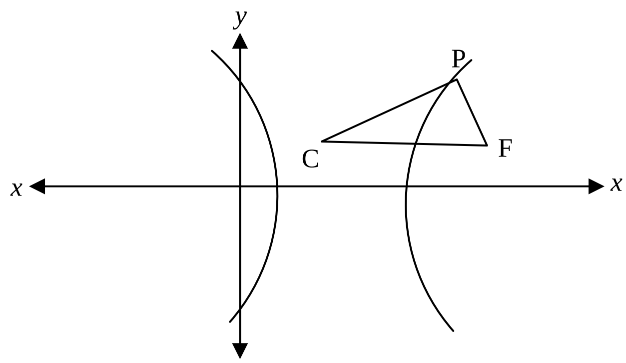
INTRODUCTION TO TRIGONOMETRY

In a previous unit, we gave a brief introduction to trigonometric relationships and how they can be used to find unknown angle measurements given two sides of a right triangle. This method was useful in determining the rotation angle of Complex numbers under multiplication. Trigonometry's applications go beyond Complex numbers and simple angle measurement. Radio waves, computer micro-circuitry and electronics, stress analysis in architecture, global navigation and sound acoustics all depend on trigonometry in one form or another. Over the course of the next few units, we will examine trigonometric relationships in more detail and learn to analyze certain real world situations according to trigonometric principles.

Basic Definitions (Right Triangle Trigonometry)

In many areas of mathematics one concept may be examined in multiple situations. In Geometry, triangles can be examined on a simple two-dimensional plane or as the face of a pyramid in three dimensions or as a system of algebraic coordinates on the *xy*-plane. Trigonometry may also be applied and examined in various mathematical settings.

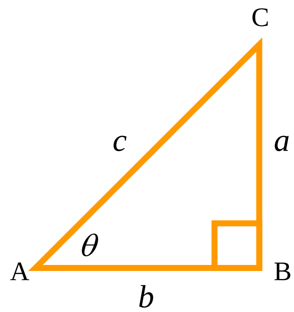
In a hyperbola, a triangle can be formed by connecting the center to a point on the hyperbola, then to a focus as in the diagram below:



The study of the sides and angles of $\triangle CPF$, and all other triangles thus formed, constitute in area of math called, "Hyperbolic Trigonometry", which has important applications to navigation, suspension bridge design and global satellite and navigation networks. Hyperbolic Trigonometry is only one form of trigonometry that can be studied and is usually explored in more detail in college level Calculus courses. For the purposes of this course, we will confine our study of trigonometry to right triangles and circles and their graphs on the *xy*-plane and on the Complex Plane.

Trigonometry: Given a right triangle $\triangle ABC$ with $\angle B = 90^\circ$, let $b = AB$, $a = BC$ and $c = AC$, be the lengths of the sides of $\triangle ABC$. For the acute angles, $\measuredangle A$ and $\measuredangle$ C, the reduced ratios: $\frac{a}{b}, \frac{a}{c}, \frac{b}{c}, \frac{b}{a}, \frac{c}{a}, \frac{c}{b}$ are unique for m$\measuredangle A$ and the m$\measuredangle C$.

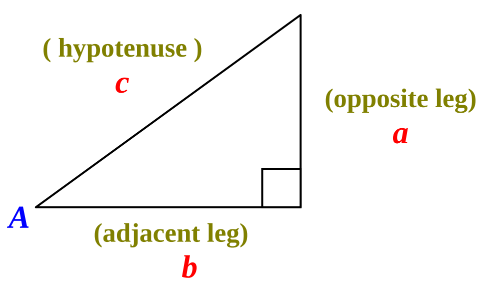
Example #1: Let m $\measuredangle A = \theta$ where $0 < \theta < 90^\circ$, then $\frac{a}{b}$, $\frac{a}{c}$, & etc., is unique to θ.



In other words, trigonometry attaches a unique real number for the ratios of the lengths of the sides of a right triangle to a measure of one of the two acute angles in the triangle in a unique way.

For any right triangle, we know that only six ratios can be formed from the lengths of the sides and that these ratios are unique to the measures of the acute angles found in the triangle. From this, we define each ratio and its relationship to the acute angles in the following manner using only angle $\measuredangle A$ as the angle to be referenced (similar ratios can be described for $\measuredangle C$).

Recall that for a right triangle the two sides that form the right angle are called the "legs" of the triangle and the third side is called the "hypotenuse". In the diagram below, definitions for the ratios of the sides of the triangle will be in reference to angle A ($\measuredangle A$ is called the "reference angle"). In this respect, the length of the leg "opposite" $\measuredangle A = a$, and the length of the leg next to or "adjacent to" $\measuredangle A = b$. The length of the hypotenuse is then equal to c.



From this, we define the following six fundamental trigonometric ratios:

Definition:	Notation
1) Sine of $\angle A = \dfrac{opposite\ leg}{hypotenuse} = \dfrac{a}{c}$	$\sin A = \dfrac{a}{c}$
2) Cosine of $\angle A = \dfrac{adjacent\ leg}{hypotenuse} = \dfrac{b}{c}$	$\cos A = \dfrac{b}{c}$
3) Tangent of $\angle A = \dfrac{opposite\ leg}{adjacent\ leg} = \dfrac{a}{b}$	$\tan A = \dfrac{a}{b}$
4) Cosecant of $\angle A = \dfrac{hypotenuse}{opposite\ leg} = \dfrac{c}{a}$	$\csc A = \dfrac{c}{a}$
5) Secant of $\angle A = \dfrac{hypotenuse}{adjacent\ leg} = \dfrac{c}{b}$	$\sec A = \dfrac{c}{b}$
6) Cotangent of $\angle A = \dfrac{adjacent\ leg}{opposite\ leg} = \dfrac{b}{a}$	$\cot A = \dfrac{b}{a}$

Example #2: Find the following trigonometric ratios for the given right triangle and referenced angle.

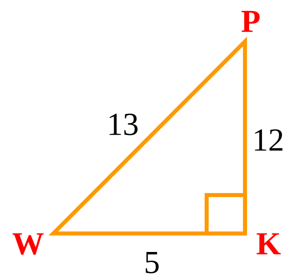
a) $\sin P$ b) $\cot P$ c) $\cos P$

For $\angle P$, the "adjacent leg" = PK = 12 and the "opposite leg" = WK = 5 with PW = 13 = "hypotenuse". Therefore:

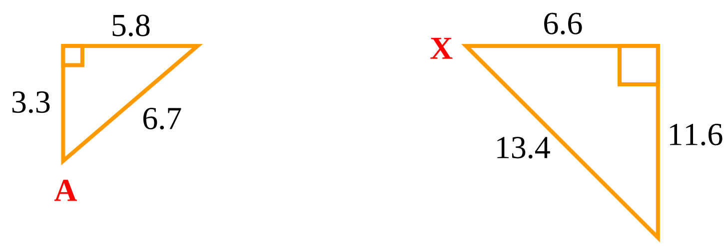
a) $\sin P = \dfrac{5}{13}$

b) $\cot P = \dfrac{12}{5}$

c) $\cos P = \dfrac{12}{13}$



Example #3: Determine if m $\measuredangle A$ = m $\measuredangle X$ in the two triangles below:



From our definition, the reduced ratio of any two sides of a right triangle is "unique" for an angle of given measure. For $\measuredangle A$ and $\measuredangle X$, we can choose any one of the six ratios defined above and compare their ratios.

Solution: find cscA and cscX

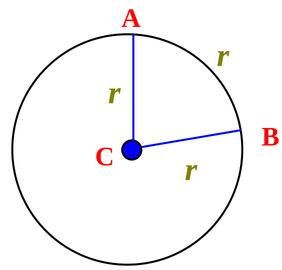
a) $\csc A = \dfrac{hypotenuse}{opposite} = \dfrac{6.7}{5.8} = 1.155$

b) $\csc X = \dfrac{hypotenuse}{opposite} = \dfrac{13.4}{11.6} = 1.155$

Therefore, we conclude that m$\measuredangle A$ = m$\measuredangle X$, even though we do not know either angle's exact measure.

Radian Measure of Angles

Traditionally angles are measured in degrees. For example, the right angle in a right triangle measures $90°$ (ninety degrees) and the degree measure of a circle is $360°$. This method for measuring angles dates back to the ancient Babylonians and is tied to their concept that there are roughly 360 days in a year. In modern times mathematicians have adopted a different method for measuring angles that ties angle measure more closely to the fundamental relationship between a circle's radius and its circumference. This method is called the "radian measure" of angles. Just as length can be measured in either yards or meters and weight can be measured in either pounds or kilograms, angles can be measured in either degrees or in radians. The radian system for angle measure is derived in the following manner:



In the circle to the right, the angle formed at the center is constructed by allowing the sides of the angle to cut the circle so that the "arc" from A to B, ($\overparen{AB}$) has the same length as the radius of the circle. By constructing $\angle ACB$ in this manner the circle is now divided into radian measure and $\angle ACB = 1^r$ (one radian "degree"). Through a process beyond the scope of this unit, the degree measure of $\angle ACB \approx 57.296^\circ$ or $1^r \approx 57.296°$.

When a circle is measured in degrees, the circle is divided into 360 equal "sectors" or 360 pie-shaped wedges. In radian measure, the circle is now divided into sectors where each sector has a degree measure of approximately $57.296°$. Dividing $\dfrac{360}{57.296} \approx 6.28$ which = $2\times(3.14) \Rightarrow 2\pi$. Therefore there are $2\pi^r$ radian angles in a circle. This provides the "fundamental conversion factor" for converting degree measure to radians or radian measure to degrees.

$$2\pi^r = 360° \quad \Rightarrow \quad \boxed{\pi^r = 180°}$$

Example #1: Convert the following degree angles to radian measure.

a) $90°$ b) $45°$ c) $60°$ d) $210°$

Solution: The fundamental conversion is $\pi^r = 180°$.

a) $90° = \dfrac{180°}{2} = \dfrac{\pi^r}{2}$

b) $45° = \dfrac{180°}{4} = \dfrac{\pi^r}{4}$

c) $60° = \frac{180°}{3} = \frac{\pi^r}{3}$

d) $210° = 180 + 30$

$= \pi + 30°$ where $30° = \frac{180°}{6} = \frac{\pi}{6}$

$= \pi + \frac{\pi}{6}$

$= \frac{7\pi}{6}$

Shortcut: $\frac{210°}{180} = \frac{7}{6}$ or $\frac{7}{6}$ of 180 = 210

$\Rightarrow \frac{7}{6}$ of π or $\frac{7\pi}{6}$

Example #2: Find the radian measures of:

a) 75° b) 245°

Solution:

a) $\frac{75}{180} = \frac{5}{12}$ =≫ 75° is $\frac{5}{12}$ of 180 = $\frac{5}{12}\pi = \frac{5\pi}{12}$

b) $\frac{245}{180} = \frac{49}{36}$ =≫ 245° is $\frac{49}{36}$ of 180 = $\frac{49}{36}\pi = \frac{49\pi}{36}$

Example #3: Convert the following radian angles to degree measure:

a) $\frac{11\pi}{6}$ b) $\frac{2\pi}{3}$ c) $\frac{13\pi}{4}$ d) $\frac{27}{5\pi}$

Solution: Recall that π^r = 180, therefore by simple substitution:

a) $\frac{11\pi}{6} = \frac{11(180)}{6} = 330°$

b) $\frac{2\pi}{3} = \frac{2(180)}{3} = 120°$

c) $\dfrac{13\pi}{4} = \dfrac{13(180)}{4} = 585°$

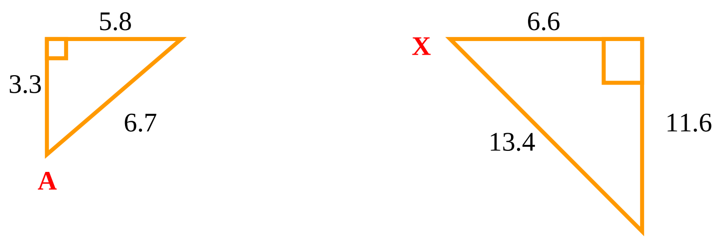
d) $\dfrac{27}{5\pi} = \dfrac{27}{5(180)} = \left.\dfrac{3}{100}\right.^{\circ} = 0.03°$

Using Trigonometric Ratios

From the definitions of trigonometric ratios, a unique angle measure is associated with the reduced ratio of any two sides of a right triangle. Mathematicians have long since determined these associations for every angle from $0°$ to $90°$ (0^r to $\frac{\pi}{2}^r$) and listed these values in tables. Recently these tables have been stored electronically in computers and in scientific calculators. For the remainder of this unit, we will use the graphing calculator and its tables of stored values for the six trigonometric ratios to find unknown angle measures or the lengths of unknown sides in right triangles.

Recall that the arctangent was used to find the rotation angles of Complex numbers.

Example #1: Determine the measure of $\angle A$ and $\angle X$ in example #2.



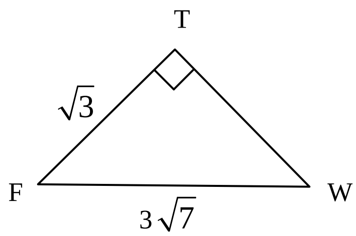
Solution: $\tan \angle A = \frac{5.8}{3.3} = 1.758 = \tan \angle X = \frac{11.6}{6.6}$

Step #1: Press [MODE] and select "Degree"

Step #2: Type [*2nd*] [tan], 1.76, [ENTER] $\approx 60.396°$

(*For these exercises we will use degree measure.*)

Example #2: Determine m$\angle W$ in the triangle below.



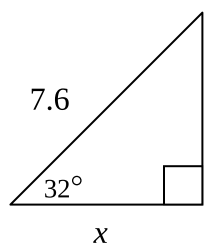
In this example $\sqrt{3}$ = opp leg and $3\sqrt{7}$ = hypotenuse for the reference angle $\measuredangle W$. In the previous example the measure of the unknown angle was determined by the arctangent. In this example the adjacent side ($\overline{TW}$) is unknown. By using the Pythagorean Theorem ,TW could be found, and then the measure of $\angle W$ determined by the arctangent.

However, the ratio $\dfrac{\sqrt{3}}{3\sqrt{7}}$ is also the $\sin\angle W$. This ratio is uniquely associated with the measure of $\angle W$ and is therefore the "arcsine" of $\angle W$ (denoted as, $\sin^{-1} W$). As with the arctangent, the arcsine and arccosines are also found on the graphing calculator.

Solution: Type [*2nd*] [sin] , $\sqrt{3}/(3\sqrt{7})$, [ENTER]

$$m\angle W \approx 12.604°$$

Example #3: Find the value of *x* in the right triangle below:



For this problem, only one side (the hypotenuse) and one acute angle in the triangle are known. The ratio $\dfrac{x}{7.6}$ must be unique for $\angle A = 32°$. For $\angle A$, the ratio $\dfrac{x}{7.6}$ represents the $\cos 32°$ (adjacent leg/hypotenuse) or $\cos 32° = \dfrac{x}{7.6}$.

Multiplying both sides by 7.6,

$$x = 7.6 \times \cos 32°$$

As mentioned before, the unique ratio for $\cos 32°$ has long been determined and is stored in the calculator. Therefore, the solution to the problem is to type the following into the calculator:

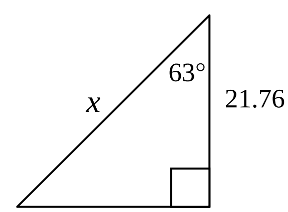
$7.6 \times \cos 32°$ [ENTER]

Answer: $x = 6.445$

*Note: Do not type [*2nd*] [cos] as this would determine an angle measure. For this problem we are finding an unknown side.

Example #4: Find the value of *x* in the right triangle below.

For this problem



$$\cos 63 = \frac{21.76}{x}$$

or $\sec 63° = \dfrac{x}{21.76} \Rightarrow 21.76 \times \sec 63° = x$

However [sec] is not found on the calculator and recall that [*2nd*] [cos] is not the reciprocal of the cosine but is the “arc cosine”. Therefore, for this problem we must first cross multiply to isolate *x* then find the answer on the graphing calculator.

Solution: $\cos 63° = \dfrac{21.76}{x} \Rightarrow x = 21.76 / \cos 63°$

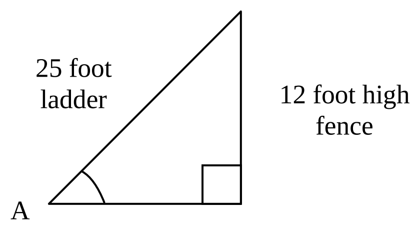
Type $21.76 \div \cos 63$, [ENTER]

$x = 47.931$

Example #5: A 25-foot ladder is leaning against the top of a 12-foot fence. What angle does the ladder make with the ground?

Solution:

Step #1: Draw a diagram to represent the situation.



Step #2: Identify the correct trigonometric ratio for $\measuredangle A$

$$\sin \measuredangle A = \frac{12}{25} \Rightarrow m\measuredangle A = \arcsin(12/25) = \sin^{-1}(12/25)$$

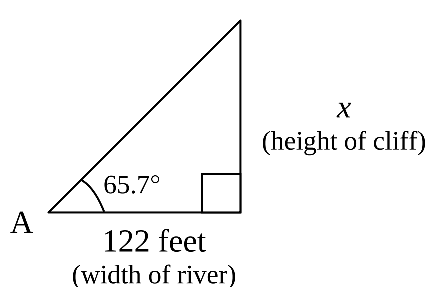
or $\boxed{2nd}$, $\sin(12/25)$ $\boxed{\text{ENTER}}$

$m\measuredangle A = 28.685°$

Example #6: A surveyor stands on one side of a river that is 122 feet wide. On the other side of the river is sheer rock cliff of unknown height. The surveyor uses a sextant and measures the angle formed by the surface of the river to the top of the cliff. He finds that this angle is $65.7°$. How high is the cliff?

Solution:

Step #1: Draw a diagram of the situation



Step #2: Choose the correct trigonometric ratio to represent the problem.

$$\tan 65.7° = \frac{x}{122}$$

Step #3: Solve for *x*.

$$122 \times \tan 65.7° = x$$

Type $122 \times \tan 65.7°$, $\boxed{\text{ENTER}}$

$$x = 270 \text{ feet}$$