

# REMAINDER AND FACTOR THEOREMS

This unit is a review of all Polynomials. In this unit you will multiply and divide polynomials using long division and synthetic division, and use the Remainder and Factor Theorems to solve problems. The second half of the unit will have you solve polynomial equations by factoring, using a graph and synthetic division and writing a polynomial function given the zeros.

Introduction to Polynomial Functions

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## Introduction to Polynomial Functions

**monomial:** a numeral, a variable, or a product of a numeral and one or more variables.

*Examples:* 5,  $x$ ,  $-3xy$

**constant:** a monomial with no variable, basically any number that has a constant value.

*Examples:* 6,  $-9$

**coefficient:** the numerical factor of a monomial.

*Examples:* in  $3xy$  the coefficient is 3, in  $\frac{x}{6}$  (also written as  $\frac{1}{6}x$ ) the coefficient is  $\frac{1}{6}$

**degree of a monomial:** the sum of the exponents of the variables.

*Example:* the monomial  $x^2yz$  (also written as  $x^2y^1z^1$ ) has a degree of 4

**polynomial:** a monomial or a sum of terms that are monomials (there can not be any variable exponents or any variables in the denominator of a term).

Binomials and trinomials are polynomials used frequently in algebra.

**binomial:** a polynomial with two terms.

*Example:*  $3x + 5$

**trinomial:** a polynomial with three terms.

*Example:*  $-4x^2 + 6x - 8$

**degree of a polynomial:** the largest monomial degree of the polynomial.

*Example:*  $9x^4 + 3x^3 + 4x^2 - x + 1$  has a degree of 4 because the highest exponent is 4.

*Example #1:* Evaluate the polynomial for  $x = 3$ .

$$-x^2 - 7x + 11$$

$$-(-3)^2 - 7(-3) + 11$$

$$-9 + 21 + 11$$

$$23$$

Replace  $x$  with  $(-3)$  and simplify.

The polynomial evaluates to 23 when  $x = 3$ .

## Evaluating Polynomial Functions

To evaluate a polynomial function for a specific value of  $x$ , replace all  $x$  variables with the value given.

*Example #1:* Evaluate  $x^3 - 2x^2 + 4x - 6$  for  $x = -2$ .

$$(-2)^3 - 2(-2)^2 + 4(-2) - 6 \quad \text{replace all } x\text{'s with } -2$$

$$(-8) - 2(4) + 4(-2) - 6 \quad \text{simplify}$$

$$-8 - 8 - 8 - 6$$

$$-30$$

The value of  $x^3 - 2x^2 + 4x - 6$  for  $x = -2$  is  $-30$ .

## Adding and Subtracting Polynomials

To add or subtract polynomials combine like terms; this means add or subtract the coefficients of any terms that have the same variable and the same exponent. Make sure the like terms are rewritten next to each other; this will make it easier to add or subtract.

When **subtracting polynomials**, make sure to **change all the signs** of the **second quantity** before you rewrite the problem.

*Example #1:*  $(6x^3 + 3x^2 - 4) + (10 - 3x - 5x^2 + 3x^3)$

$$6x^3 + 2x^3 + 3x^2 - 5x^2 - 3x - 4 + 10$$

arrange the terms in descending powers

$$8x^3 - 2x^2 - 3x + 6$$

collect like terms

*Example #2:*  $(5x^2 - 6x - 11) - (-8x^3 + x^2 + 2)$

$$5x^2 - 6x - 11 + 8x^3 - x^2 - 2$$

\*notice that all the signs have changed on the second quantity.

$$8x^3 + 5x^2 - x^2 - 6x - 11 - 2$$
$$8x^3 + 4x^2 - 6x - 13$$

## Products and Factors of Polynomials

To write a function in standard form means to multiply all terms together and write them in descending order (largest to smallest).

*Example #1:*  $f(x) = (x-1)(x+4)(x-3)$

$$f(x) = (x^2 + 3x - 4)(x-3) \quad \text{FOIL the first two quantities}$$

$$f(x) = x^2(x-3) + 3x(x-3) - 4(x-3) \quad \text{use the distributive property to multiply}$$

$$f(x) = x^3 - 3x^2 + 3x^2 - 9x - 4x + 12 \quad \text{combine like terms}$$

$$f(x) = x^3 - 13x + 12$$

## Factoring Polynomials

Just as a quadratic expression is factored by writing it as a product of two factors, a polynomial expression of a degree greater than 2 is factored by writing it as a product of more than two factors.

*Example #2:*  $x^3 - 10x^2 + 16x$  factor out the GCF of  $x$

$$x(x^2 - 10x + 16) \quad \text{factor the trinomial into binomials}$$

$$x(x-8)(x-2)$$

*Example #3:*  $x^3 + 6x^2 - 5x - 50$  this polynomial can be factored in pairs

$$(x^3 + 6x^2) + (-5x - 50) \quad \text{group the terms in pairs}$$

$$x^2(x+6) - 5(x+6) \quad \text{factor out the GCF of each pair to produce a quantity that is common}$$

$$(x+6)(x^2 - 5) \quad \text{factor } (x+6) \text{ from each term}$$

## Factoring the Sum and Difference of Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

*Example #4:* Factor  $x^3 + 27$ .

$$(x + 3)(x^2 - 3x + 9)$$

The  $x$  in the first quantity comes from  $\sqrt[3]{x^3}$ .

The 3 in the first quantity comes from  $\sqrt[3]{27}$ .

The  $x^2$  in the second quantity comes from squaring  $x$  in the first quantity.

The  $3x$  in the second quantity comes from multiplying both terms in the first quantity.

The 9 in the second quantity comes from squaring the 3 in the first quantity.

The signs for the sum of cubes will always be  $(+, - +)$ .

*Example #5:* Factor  $x^3 - 125$ .

$$(x - 5)(x^2 + 5x + 25)$$

The signs for the difference of cubes will always be  $(-, + +)$ .

## Long Division of Polynomials

To use long division:

- 1) Make sure that all terms are in descending order.
- 2) Make sure that all terms are represented, for example, if there is no  $x^2$  term in the polynomial insert  $0x^2$  for that term that is missing.

*Example #1:*  $\frac{x^3 + 3x^2 - 4x - 12}{x - 2}$

$$\begin{array}{r}
 x^2 + 5x + 6 \\
 x - 2 \overline{) x^3 + 3x^2 - 4x - 12} \\
 \underline{-(x^3 - 2x^2)} \quad \leftarrow \text{multiply } x - 2 \text{ by } x^2 \text{ and subtract} \\
 5x^2 - 4x \quad \leftarrow \text{bring down the } -4x, \\
 \underline{-(5x^2 - 10x)} \quad \leftarrow \text{how many times will } x \text{ go into } 5x^2 \\
 6x - 12 \quad \leftarrow \text{multiply } x - 2 \text{ by } 5x \text{ and subtract} \\
 \underline{6x - 12} \quad \leftarrow \text{how many times will } x \text{ go into } 6x \\
 0
 \end{array}$$

*Example #2:*  $(x^3 - 7x - 6) \div (x + 1)$

since there is no  $x^2$  term, replace it with  $0x^2$  to hold the place value

$$\begin{array}{r}
 x^2 - x - 6 \\
 x + 1 \overline{) x^3 + 0x^2 - 7x - 6} \\
 \underline{-(x^3 + x^2)} \quad \leftarrow \text{multiply } x + 1 \text{ by } x^2 \text{ and subtract} \\
 -x^2 - 7x \quad \leftarrow \text{bring down the } -7x \\
 \underline{-(x^2 - x)} \quad \leftarrow \text{how many times will } x \text{ go into } -x^2 \\
 -6x - 6 \quad \leftarrow \text{multiply } x + 1 \text{ by } -x \text{ and subtract} \\
 \underline{-6x - 6} \quad \leftarrow \text{how many times will } x \text{ go into } -6x \\
 0
 \end{array}$$

## Synthetic Division

Another form of division is called synthetic division. Synthetic division can be used to divide a polynomial only by a linear binomial of the form  $x - r$  and only uses the coefficients of each term. When using nonlinear divisors, long division must be used.

*Example #1:* Use synthetic division to solve:  $(x^3 + 3x^2 - 4x - 12) \div (x - 2)$

*Step #1:* Write out the coefficients of the polynomial, and then write the  $r$ -value, 2, of the divisor  $x - 2$ . **Notice that you use the opposite of the divisor sign.** Write the first coefficient, 1, below the line.

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -4 & -12 \\ & & & & \\ \hline & 1 & & & \end{array}$$

*Step #2:* Multiply the  $r$ -value, 2, by the number below the line, 1, and write the product, 2, below the next coefficient.

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -4 & -12 \\ & & 2 & & \\ \hline & 1 & & & \end{array}$$

*Step #3:* Write the **sum** (not the difference) of 3 and 2, (5), below the line. Multiply 2 by the number below the line, 5, and write the product, 10, below the next coefficient.

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -4 & -12 \\ & & 2 & 10 & \\ \hline & 1 & 5 & & \end{array}$$

*Step #4:* Write the sum of  $-4$  and 10, (6), below the line. Multiply 2 by the number below the line, 6, and write the product, 12, below the next coefficient.

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -4 & -12 \\ & & 2 & 10 & 12 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

The remainder is 0, and the resulting numbers 1, 5 and 6 are the coefficients of the quotient. The quotient will start with an exponent that is one less than the dividend.



The result is  $x^2 + 5x + 6$ .

Thus,  $(x^3 + 3x^2 - 4x - 12) \div (x - 2) = x^2 + 5x + 6$ .

Now let's try one that has a remainder.

*Example #2:*  $(x^3 - 2x^2 - 22x + 40) \div (x - 4)$

$$\begin{array}{r|rrrr} 4 & 1 & -2 & -22 & 40 \\ & & 4 & 8 & -56 \\ \hline & 1 & 2 & -14 & \underline{-16} \end{array}$$

Since there is a remainder in this problem, the answer is written using a fraction with the divisor,  $x - 4$ , as the denominator. Notice that the sign between the last term and the fraction is the same as the sign of the remainder.

Answer:  $x^2 + 2x - 14 - \frac{16}{x - 4}$

## Factor Theorem

**Factor Theorem:**  $x - r$  is a factor of the polynomial expression that defines the function  $P$ , if and only if,  $r$  is a solution of  $P(x) = 0$ ; that is, if and only if  $P(r) = 0$ .

What this is saying is that a binomial is a factor of a polynomial expression, if and only if, the numerical value  $r$ , when replaced for all the  $x$  values, yields 0.

*Example #1:*  $x + 2$  is a factor of  $x^3 - 2x^2 - 5x + 6$ , if and only if,  $f(-2) = 0$ .

To find  $f(-2)$  replace all  $x$  values with  $-2$ . Notice this is the opposite of the value given for the factor  $x + 2$ .

$$\begin{aligned} f(-2) &= (-2)^3 - 2(-2)^2 - 5(-2) + 6 && \text{replace all } x \text{ values with } -2 \\ &= -8 - 8 + 10 + 6 \\ &= -16 + 16 \\ &= 0 \end{aligned}$$

This means that  $x + 2$  is a factor of the polynomial because  $f(-2) = 0$ .

*Example #2:* Is  $x - 1$  a factor of  $f(x) = x^3 - x^2 - 5x - 3$ ?

$$\begin{aligned} f(1) &= (1)^3 - (1)^2 - 5(1) - 3 && \text{replace all } x \text{ values with } 1 \\ &= 1 - 1 - 5 - 3 \\ &= -8 \end{aligned}$$

This means that  $x - 1$  is not a factor of the polynomial because  $f(1) \neq 0$ .

## Solving Polynomial Equations by Factoring

To solve a polynomial equation means to find the roots or zeros of the function. The roots or zeros is where the polynomial crosses the  $x$ -axis.

To factor:

- look for a GCF
- look for a difference of squares  $a^2 - b^2$
- use trial and error

*Example #1:*  $x^2 + 4x - 21 = 0$

there is no GCF and the leading coefficient is 1, so find factors of  $-21$  that add up to  $+4$

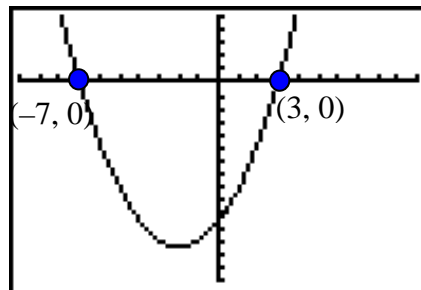
$$(x + 7)(x - 3) = 0$$

set each of these factors = to 0

$$x + 7 = 0 \quad x - 3 = 0$$

$$x = -7 \quad x = 3$$

This means the polynomial  $x^2 + 4x - 21 = 0$  has 2 real zeros,  $-7$  and  $3$ , the two places it crosses the  $x$ -axis. The graph below illustrates this.



*Example #2:*  $5x^3 - 12x^2 + 4x = 0$

$$x(5x^2 - 12x + 4) = 0$$

GCF is  $x$

$$x(5x - 2)(x - 2) = 0$$

Trial and Error

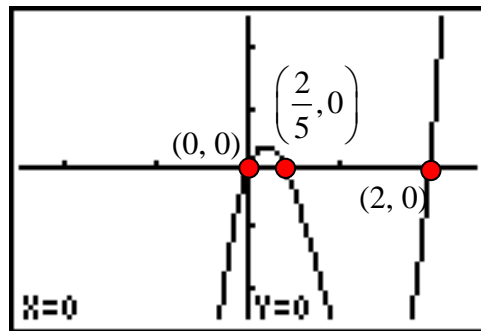
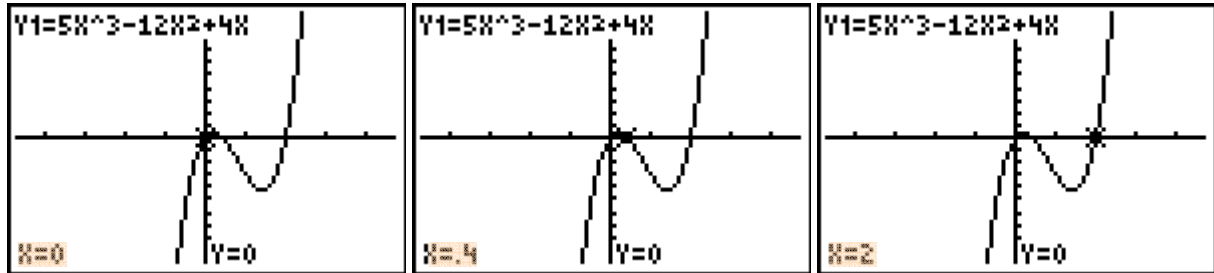
$$x = 0, \quad 5x - 2 = 0, \quad x - 2 = 0$$

Set each factor = to 0 and solve.

$$x = 0, \quad x = \frac{2}{5}, \quad x = 2$$

This means that the polynomial  $5x^3 - 12x^2 + 4x = 0$  has 3 real roots,  $0$ ,  $\frac{2}{5}$ , and  $2$ .

This is where the polynomial crosses the  $x$ -axis.

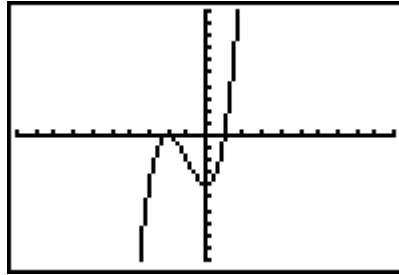


## Solving Polynomial Equations by Using a Graph and Synthetic Division

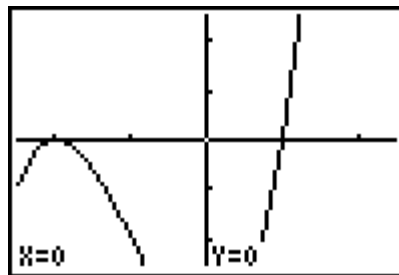
To solve a polynomial function by graphing and using synthetic division:

- 1) Graph the function on your calculator.

$$y = x^3 + 3x^2 - 4$$



- 2) Determine where the graph crosses the  $x$ -axis. In this case the graph looks like it touches the  $x$ -axis at  $(-2, 0)$ . **\*Make sure you are viewing the standard viewing screen on your calculator, ZOOM 6. Then press ZOOM2 ENTER to zoom in and get a better look.**



- 3) Use synthetic division to test your choice. **Make sure to insert a 0 to hold the place of the missing  $x$  term.**

$$y = x^3 + 3x^2 + 0x - 4$$

$$\begin{array}{r|rrrr}
 -2 & 1 & 3 & 0 & -4 \\
 & & -2 & -2 & 4 \\
 \hline
 & 1 & 1 & -2 & 0
 \end{array}$$

- 4) Since the remainder is 0, we now know that  $x + 2$  is a factor of  $x^3 + 3x^2 - 4 = 0$  and the other factor is  $x^2 + x - 2$  (coefficients determined above). Write the polynomial in factored form, and then factor  $x^2 + x - 2$ .

$$x^3 + 3x^2 - 4 = 0$$

$$(x + 2)(x^2 + x - 2) = 0 \quad \text{Factor completely.}$$

$$(x + 2)(x + 2)(x - 1) = 0 \quad \text{Use the zero product property to solve.}$$

$$x + 2 = 0 \quad x + 2 = 0 \quad x - 1 = 0$$

$$x = -2 \quad x = -2 \quad x = 1$$

Therefore the roots of this polynomial are 1 and  $-2$ , with  $-2$  occurring twice.

### **Multiplicity of Roots**

**Multiplicity** is the number of times a zero occurs in a function or the number of times a root occurs in an equation.

$$(x + 2)(x + 2)(x - 1) = 0 \text{ can be written as } (x + 2)^2(x - 1) = 0$$

\*The root of 1 is called a “simple root” (or has a multiplicity of 1) and the root of  $-2$  is a root of multiplicity 2.

## Solving Polynomial Equations by Using Variable Substitution and Factoring

If a polynomial has a degree of more than 2, you can use variable substitution to solve.

*Example #1:* Solve for  $x$ :  $x^4 - 9x^2 + 14 = 0$

*Step #1:* Substitute  $u$  for  $x^2$ .

$$(x^2)^2 - 9(x^2) + 14 = 0$$

$$u^2 - 9u + 14 = 0$$

*Step #2:* Factor and solve.

$$(u - 7)(u - 2) = 0$$

$$u - 7 = 0 \quad u - 2 = 0$$

$$u = 7 \quad u = 2$$

*Step #3:* Replace  $u$  with  $x^2$ .

$$x^2 = 7 \quad x^2 = 2$$

$$x = \pm\sqrt{7} \quad x = \pm\sqrt{2}$$

*Example #2:* Solve for  $z$ :  $z^5 - 28z^3 + 27z = 0$

$$z^5 - 28z^3 + 27z = 0 \quad \text{factor out a GCF of } z$$

$$z(z^4 - 28z^2 + 27) = 0 \quad \text{substitute } u \text{ for } z^2$$

$$z(u^2 - 28u + 27) = 0 \quad \text{factor}$$

$$z(u - 27)(u - 1) = 0 \quad \text{solve for } u$$

$$z = 0 \quad u - 27 = 0 \quad u - 1 = 0$$

$$u = 27 \quad u = 1 \quad \text{replace } u \text{ with } z^2$$

$$z = 0 \quad z^2 = 27 \quad z^2 = 1 \quad \text{solve for } z$$

$$z = 0, \quad z = \pm\sqrt{27}, \quad z = \pm\sqrt{1} = \pm 1$$

Therefore, the roots of the polynomial  $z^5 - 28z^3 + 27z = 0$  would be 0, 1, -1,  $+\sqrt{27}$ , and  $-\sqrt{27}$ .



## Rational Root Theorem

### Rational Root Theorem

Let  $P$  be a polynomial function with integer coefficients in standard form.

If  $\frac{p}{q}$  (in lowest terms) is a root of  $P(x) = 0$ , then

$p$  is a factor of the constant term of  $P$  and

$q$  is a factor of the leading coefficient of  $P$

*Example #1:* Find all of the rational roots of  $8x^3 + 10x^2 - 11x + 2 = 0$ .

According to the Rational Root Theorem,  $\frac{p}{q}$  is a root of  $8x^3 + 10x^2 - 11x + 2 = 0$  if  $p$  is a factor of the constant term, 2, and  $q$  is a factor of the leading coefficient, 8.

*Step #1:* Make an organized list of all factors of 2 and 8.

factors of 2:  $\pm 1, \pm 2$

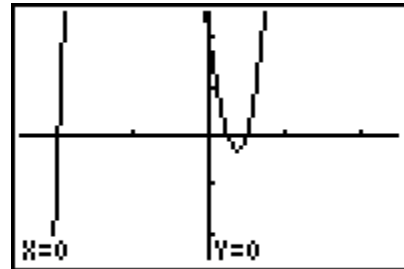
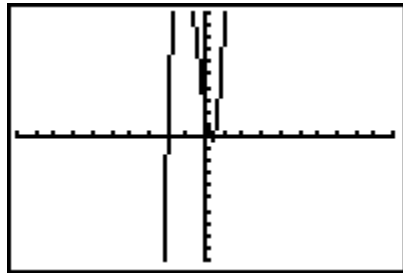
factors of 8:  $\pm 1, \pm 2, \pm 4, \pm 8$

*Step #2:* Form all quotients that have factors of 2 in the numerator and factors of 8 in the denominator.

$\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}$	$\pm \frac{2}{1}, \pm \frac{2}{2}, \pm \frac{2}{4}, \pm \frac{2}{8}$
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\*Some rational numbers are repeated, such as  $\pm \frac{1}{4}$  and  $\pm \frac{2}{8}$ .

Step #3: Graph the function on your graphing calculator. Use the button that is marked  $X\T\theta n$  for the variable  $x$ . Also to raise a number to the third power use the  $\square^{\wedge}$  button that is located above the  $\square\div$  key. The standard viewing screen is the figure on the left. ZOOMIN ENTER is the figure on the right.



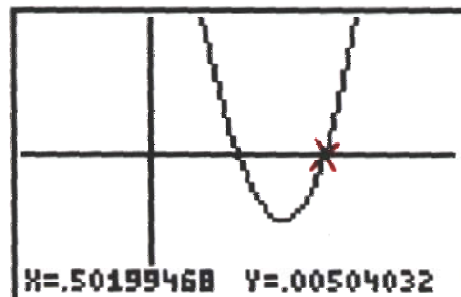
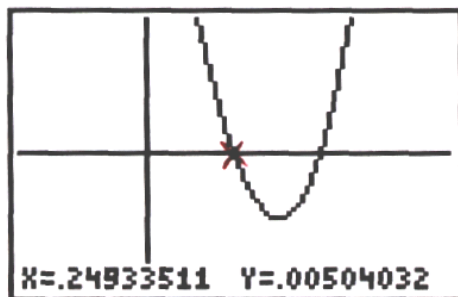
One root appears to be  $-2$ . Test whether  $P(-2) = 0$  by using synthetic division.

$$\begin{array}{r|rrrr}
 -2 & 8 & 10 & -11 & 2 \\
 & & -16 & 12 & -2 \\
 \hline
 & 8 & -6 & 1 & 0
 \end{array}$$

← remainder

Since the remainder is 0, this means that  $P(-2) = 0$  is true and that  $-2$  is a root of the polynomial.

Step #4: From the graph there appears to be two real zeros between 0 and 1. Zoom in ( $\square$ ZOOM $\square$  2) and use the trace feature to check. Press  $\square$ TRACE $\square$  and move the cursor around with the  $\square$  $\leftarrow$   $\square$  $\rightarrow$  arrow keys until the y-value on your screen gets very very close to 0.  $\square$ ZOOM $\square$ , 2: Zoom In,  $\square$ ENTER $\square$  if necessary to locate this point.



It looks like there are two real roots, one at  $\frac{1}{4}$  and the other at  $\frac{1}{2}$ . Test whether

$P\left(\frac{1}{4}\right)=0$  and  $P\left(\frac{1}{2}\right)=0$  by using synthetic division.

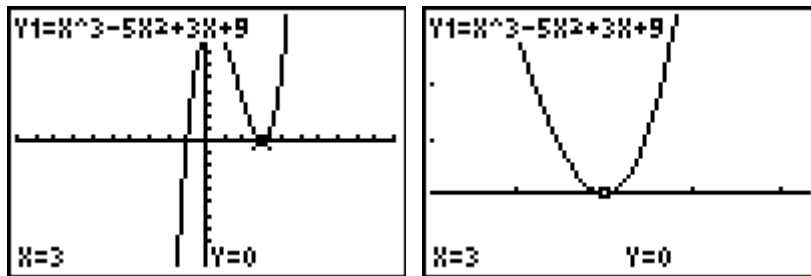
$$\begin{array}{r|rrrr} \frac{1}{4} & 8 & 10 & -11 & 2 \\ & & 2 & 3 & -2 \\ \hline & 8 & 12 & -8 & 0 \end{array} \qquad \begin{array}{r|rrrr} \frac{1}{2} & 8 & 10 & -11 & 2 \\ & & 4 & 7 & -2 \\ \hline & 8 & 14 & -4 & 0 \end{array}$$

Since  $P\left(\frac{1}{4}\right)=0$  and  $P\left(\frac{1}{2}\right)=0$ ,  $\frac{1}{4}$  and  $\frac{1}{2}$  are also roots of the polynomial.

Therefore, there are three roots  $-2$ ,  $\frac{1}{4}$ , and  $\frac{1}{2}$ .

\*When a polynomial function  $P(x)$  touches but does not cross the  $x$ -axis at  $(r, 0)$ , then  $P(x) = 0$  has a double root.

For example:  $P(x) = (x - 3)(x - 3)(x + 1)$  or  $P(x) = x^3 - 5x^2 + 3x + 9$  touches the  $x$ -axis at  $(3, 0)$  but does not cross the  $x$ -axis there, as shown in the graphs below.



Therefore, 3 is a **double root** of  $P(x) = 0$ .