SOLVING POLYNOMIAL EQUATIONS

In this unit you will solve polynomial equations by factoring and using a graph with synthetic division. You will also find the real zeros of polynomial functions and state the multiplicity of each. The unit concludes by using the Rational Root Theorem to find zeros of a polynomial function. Finally, the Fundamental Theorem is used to write a polynomial function given sufficient information about its zeros.

<div align="center">

Solving Polynomial Equations by Factoring

Solving Polynomial Equation by Using a Graph and Synthetic Division

Solving Polynomial Equations by Using Variable Substitution and Factoring

Rational Root Theorem

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Solving Polynomial Equations by Factoring

To solve a polynomial equation means to find the roots or zeros of the function. The roots or zeros is where the polynomial crosses the x-axis.

To factor:
a.) look for a GCF
b.) look for a difference of squares $a^2 - b^2$
c.) use trial and error

$Example~\#1$: $x^2 + 4x - 21 = 0$ there is no GCF and the leading coefficient is 1, so find factors of –21 that add up to +4
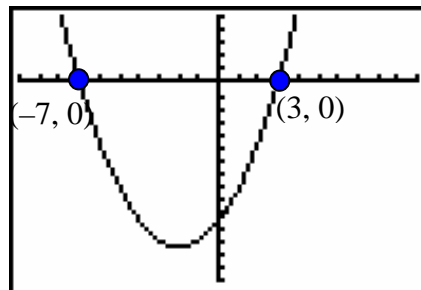
$(x + 7)(x - 3) = 0$ set each of these factors = to 0

$x + 7 = 0$ $x - 3 = 0$

$x = -7$ $x = 3$

This means the polynomial $x^2 + 4x - 21 = 0$ has 2 real zeros, –7 and 3, the two places it crosses the x-axis. The graph below illustrates this.

$Example~\#2$: $5x^3 - 12x^2 + 4x = 0$

$x(5x^2 - 12x + 4) = 0$ GCF is x
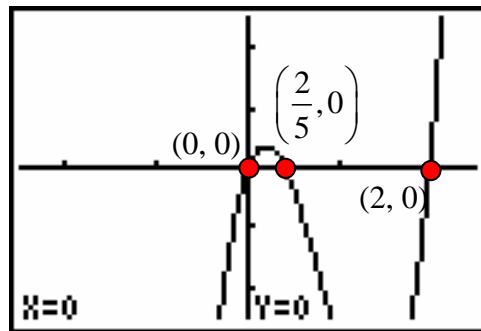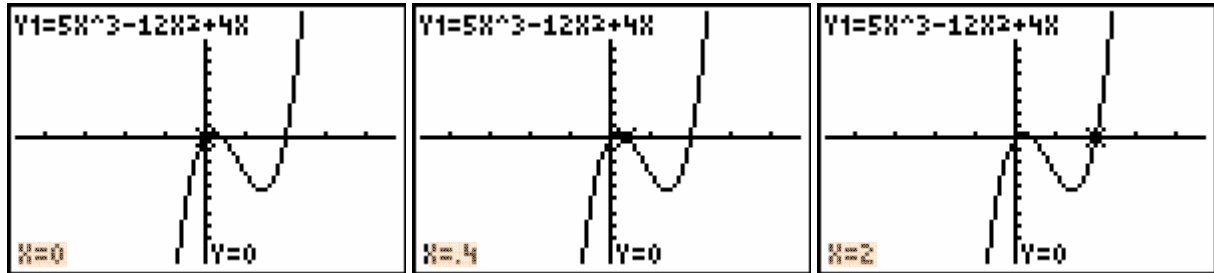
$x(5x - 2)(x - 2) = 0$ Trial and Error

$x = 0,~~5x - 2 = 0,~x - 2 = 0$ Set each factor = to 0 and solve.

$x = 0,~x = \dfrac{2}{5},~x = 2$

This means that the polynomial $5x^3 - 12x^2 + 4x = 0$ has 3 real roots, 0, $\dfrac{2}{5}$, and 2.
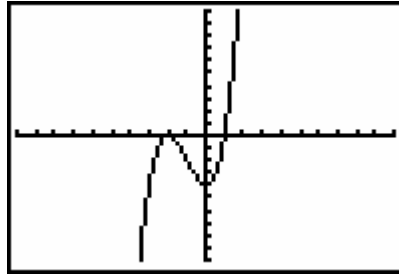
This is where the polynomial crosses the x-axis.

## Solving Polynomial Equations by Using a Graph and Synthetic Division

To solve a polynomial function by graphing and using synthetic division:

1) Graph the function on your calculator.

$$y = x^3 + 3x^2 - 4$$

2) Determine where the graph crosses the x-axis. In this case the graph looks like it touches the x-axis at (–2, 0). ***Make sure you are viewing the standard viewing screen on your calculator, ZOOM 6. Then press ZOOM2 ENTER to zoom in and get a better look.**
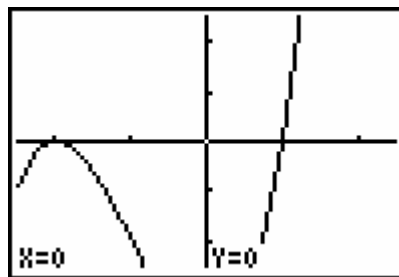
3) Use synthetic division to test your choice. **Make sure to insert a 0 to hold the place of the missing x term.** $y = x^3 + 3x^2 + 0x - 4$

$$
\begin{array}{r|rrrr}
-2 & 1 & 3 & 0 & -4 \\
   &   & -2 & -2 & 4 \\
\hline
   & 1 & 1 & -2 & \boxed{0}
\end{array}
$$

4) Since the remainder is 0, we now know that $x + 2$ is a factor of $x^3 + 3x^2 - 4 = 0$ and the other factor is $x^2 + x - 2$ (coefficients determined above). Write the polynomial in factored form, and then factor $x^2 + x - 2$.

$$x^3 + 3x^2 - 4 = 0$$

$$(x+2)(x^2 + x - 2) = 0 \qquad \text{Factor completely.}$$

$$(x + 2)(x +2)(x - 1) = 0 \qquad \text{Use the zero product property to solve.}$$

$x + 2 = 0$	$x + 2 = 0$	$x - 1 = 0$
$x = -2$	$x = -2$	$x = 1$

Therefore the roots of this polynomial are 1 and –2, with –2 occurring twice.

Multiplicity of Roots

Multiplicity is the number of times a zero occurs in a function or the number of times a root occurs in an equation.

$(x+2)(x+2)(x-1) = 0$ can be written as $(x+2)^2(x-1) = 0$

*The root of 1 is called a "simple root" (or has a multiplicity of 1) and the root of –2 is a root of multiplicity 2.

Solving Polynomial Equations by Using Variable Substitution and Factoring

If a polynomial has a degree of more than 2, you can use variable substitution to solve.

Example #1: Solve for x: $x^4 - 9x^2 + 14 = 0$

Step #1: Substitute u for x^2.

$$(x^2)^2 - 9(x^2) + 14 = 0$$

$$u^2 - 9u + 14 = 0$$

Step #2: Factor and solve.

$$(u - 7)(u - 2) = 0$$

$$u - 7 = 0 \qquad u - 2 = 0$$

$$u = 7 \qquad u = 2$$

Step #3: Replace u with x^2.

$$x^2 = 7 \qquad x^2 = 2$$

$$x = \pm\sqrt{7} \qquad x = \pm\sqrt{2}$$

Example #2: Solve for z: $z^5 - 28z^3 + 27z = 0$

$$z^5 - 28z^3 + 27z = 0 \qquad \text{factor out a GCF of } z$$

$$z(z^4 - 28z^2 + 27) = 0 \qquad \text{substitute } u \text{ for } z^2$$

$$z(u^2 - 28u + 27) = 0 \qquad \text{factor}$$

$$z(u - 27)(u - 1) = 0 \qquad \text{solve for } u$$

$$z = 0 \quad u - 27 = 0 \qquad u - 1 = 0$$

$$u = 27 \qquad u = 1 \qquad \text{replace } u \text{ with } z^2$$

$$z = 0 \qquad z^2 = 27 \qquad z^2 = 1 \qquad \text{solve for } z$$

$$z = 0, \quad z = \pm\sqrt{27}, z = \pm\sqrt{1} = \pm 1$$

Therefore, the roots of the polynomial $z^5 - 28z^3 + 27z = 0$ would be 0, 1, –1, $+\sqrt{27}$, and $-\sqrt{27}$.

Rational Root Theorem

Rational Root Theorem

Let P be a polynomial function with integer coefficients in standard form.

If $\dfrac{p}{q}$ (in lowest terms) is a root of $P(x) = 0$, then

      p is a factor of the constant term of P and

      q is a factor of the leading coefficient of P

Example #1: Find all of the rational roots of $8x^3 + 10x^2 - 11x + 2 = 0$.

According to the Rational Root Theorem, $\dfrac{p}{q}$ is a root of $8x^3 + 10x^2 - 11x + 12 = 0$ if p is a factor of the constant term, 2, and q is a factor of the leading coefficient, 8.

    Step #1: Make an organized list of all factors of 2 and 8.

        factors of 2: ± 1, ± 2

        factors of 8: ± 1, ± 2, ± 4, ± 8

    Step #2: Form all quotients that have factors of 2 in the numerator and factors of 8 in the denominator.

$\pm\dfrac{1}{1}$, $\pm\dfrac{1}{2}$, $\pm\dfrac{1}{4}$, $\pm\dfrac{1}{8}$	$\pm\dfrac{2}{1}$, $\pm\dfrac{2}{2}$, $\pm\dfrac{2}{4}$, $\pm\dfrac{2}{8}$

*Some rational numbers are repeated, such as $\pm\dfrac{1}{4}$ and $\pm\dfrac{2}{8}$.

Step #3: Graph the function on your graphing calculator. Use the button that is marked *XTΘn* for the variable *x*. Also to raise a number to the third power use the $\boxed{\wedge}$ button that is located above the $\boxed{\div}$ key. The standard viewing screen is the figure on the left. ZOOMIN ENTER is the figure on the right.
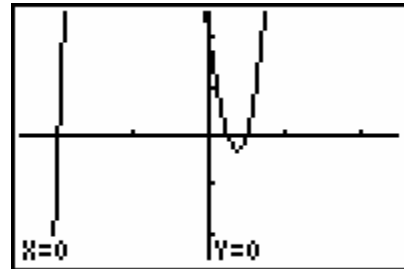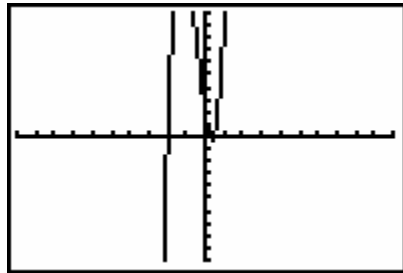

One root appears to be –2. Test whether $P(-2) = 0$ by using synthetic division.

$$
\begin{array}{r|rrrr}
-2 & 8 & 10 & -11 & 2 \\
   &   & -16 & 12 & -2 \\
\hline
   & 8 & -6 & 1 & \boxed{0}
\end{array}
$$

remainder

Since the remainder is 0, this means that $P(-2) = 0$ is true and that –2 is a root of the polynomial.

Step #4: From the graph there appears to be two real zeros between 0 and 1. Zoom in ($\boxed{\text{ZOOM}}$ 2) and use the trace feature to check. Press $\boxed{\text{TRACE}}$ and move the cursor around with the $\boxed{\langle}$ $\boxed{\rangle}$ arrow keys until the *y*-value on your screen gets very very close to 0. $\boxed{\text{ZOOM}}$, 2: Zoom In, $\boxed{\text{ENTER}}$ if necessary to locate this point.
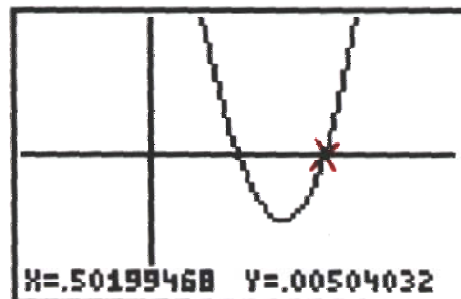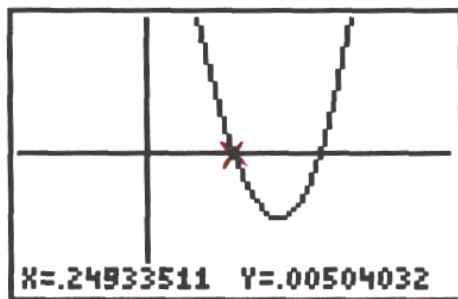
It looks like there are two real roots, one at $\dfrac{1}{4}$ and the other at $\dfrac{1}{2}$. Test whether $P\left(\dfrac{1}{4}\right) = 0$ and $P\left(\dfrac{1}{2}\right) = 0$ by using synthetic division.
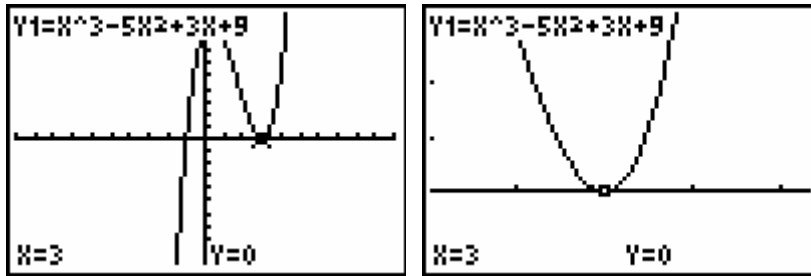
$\dfrac{1}{4}$	8	10	−11	2
		2	3	−2
	8	12	−8	0

$\dfrac{1}{2}$	8	10	−11	2
		4	7	−2
	8	14	−4	0

Since $P\left(\dfrac{1}{4}\right) = 0$ and $P\left(\dfrac{1}{2}\right) = 0$, $\dfrac{1}{4}$ and $\dfrac{1}{2}$ are also roots of the polynomial.

Therefore, there are three roots −2, $\dfrac{1}{4}$, and $\dfrac{1}{2}$.

*When a polynomial function $P(x)$ touches but does not cross the x-axis at $(r, 0)$, then $P(x) = 0$ has a double root.

For example: $P(x) = (x - 3)(x - 3)(x+1)$ or $P(x) = x^3 - 5x^2 + 3x + 9$ touches the x-axis at $(3, 0)$ but does not cross the x-axis there, as shown in the graphs below.

Therefore, 3 is a **double root** of $P(x) = 0$.