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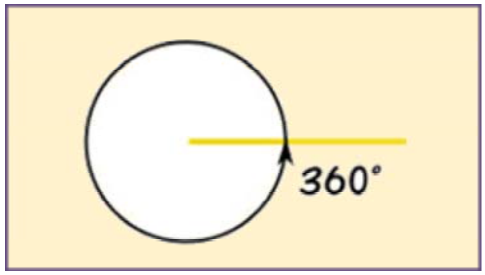


ROTATIONAL MOTION

Unit Overview

During this unit, we will discuss velocity and acceleration of different points moving along a circular, rotating object. Later in the unit we will also see how forces affect objects moving in a circular motion to create acceleration.

Rotational Motion, Angular Quantities

During this chapter we will be referring to objects that travel in a circular motion as being a **rigid rotating object**. We call it a rigid object because we are assuming that its shape is never changing. Sometimes when forces act on an object, the shape can be deformed, but most of the time it is so small that it will not affect the overall system of the rotating object.

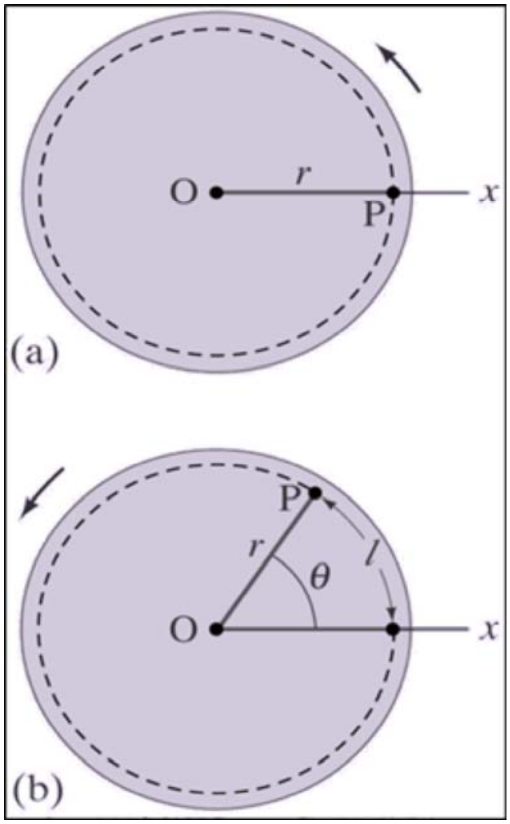
During rotational motion, all points on the object move in circles around the **axis of rotation.** The axis of rotation is referring to the center of the circle. The radius of the circle will be written as the symbol *r*. Any point along a circle with the same radius will sweep out equal angles in the same amount of time. In physics it is common to refer to an angle as a radian when talking about rotational motion.

A **radian** is defined as the angle subtended by an arc whose length is equal to the radius. If the arc length and the radius are the same, that is equal to 1 radian. This yields the equation:

$$\theta = \frac{l}{r}$$

where *l* is the arc length.

The picture below shows a point "P" on a circle rotating around the axis of rotation "O", the distance along the arc of the circle that "P" travels is the arc length "l"



When talking about the radian, it does not have any units because it is a ratio of two lengths. But when given an angle in radians, it is best to write *rad* after it to remind us that it is not in degrees. Radians are however related to degrees.

A complete circle has 360 degrees, which is the same as an arc length equaling the circumference of a circle, so $l = 2\pi r^2$ and since $\theta = l/r$ we can substitute $2\pi r^2$ in for l and get 2π rad in a complete circle, so **360° = 2π rad** ***One radian is then equal to 360/2π = 57.3°*** An object that has completed one complete revolution has rotated through 360° or 2π rad.

Examples:
1. A car tire rotates 6.7 revolutions. How many radians has it rotated?
Knowing that 1 revolution = 360 degrees = 2π rad, we can just convert to the units needed

6.7 revolutions $(2\pi \text{ rad} / 1 \text{ rev}) = 13.4\pi \text{ rad} =$ **42.1 rad**

OR

6.7 (360) / 57.3 = **42.1 rad** **-** knowing that there are 57.3 degrees per radian, you can simply divide the total degrees traveled along the circle and divide it by 57.3 degree.

A bird's eye can distinguish objects that have an arc angle no smaller than 3×10^{-4} rad.

a) How many degrees is this?
b) How small of an object can the bird distinguish when flying at a height of 100m?

c) Convert 3×10^{-4} radians to degrees

3×10^{-4} $(360°/2\pi \text{ rad}) = .017°$

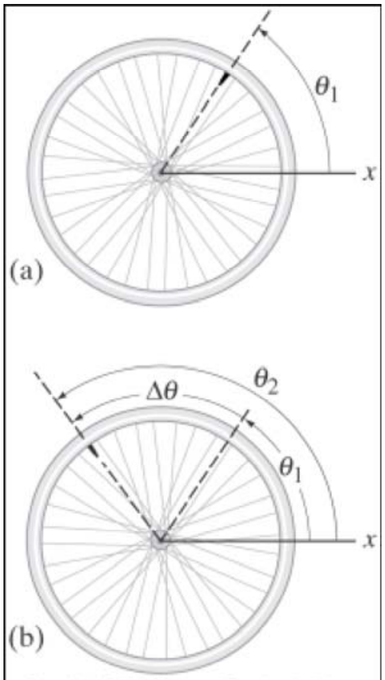
d) In this problem, we know that the radius is 100m and the θ is 3×10^{-4} radians we can use the equation $\theta = l/r$ so $3 \times 10^{-4}(100) = .03$ m or **3 cm.**

Angular velocity and acceleration

Velocity and acceleration for a rotating object mean the same thing as it did in unit 2 for an object moving along a linear path. For angular velocity, because it is now angular displacement divided by time, we will use the change in angular displacement which is where θ_1 is the initial angle and θ_2 is the final angle.

$$\boxed{\Delta\theta = \theta_2 - \theta_1}$$

The picture below shows how the change in angular displacement is found.



The **angular velocity** will be written with the Greek lowercase omega symbol, ω. The equation for angular velocity is then written as

$$\omega = \Delta\theta / \Delta t$$

Angular velocity has units of radians/second (rad/s). One thing to remember is that all points along a rigid rotating object will rotate with the same angular velocity, because every position on the object moves through the same angle per same amount of time.

Just like linear velocity, an object can have a negative velocity if traveling west or south. With angular velocity, an object is considered *positive* if it is moving in a *counterclockwise* direction and *negative* if it is moving *clockwise.*

Angular acceleration is written with the Greek lowercase letter alpha (α). Angular acceleration is the change in angular velocity divided by the amount of time needed to make this change. Which gives us the equation:

$$\alpha = \omega_2 - \omega_1 / \Delta t \qquad \text{OR} \qquad \alpha = \Delta\omega/\Delta t$$

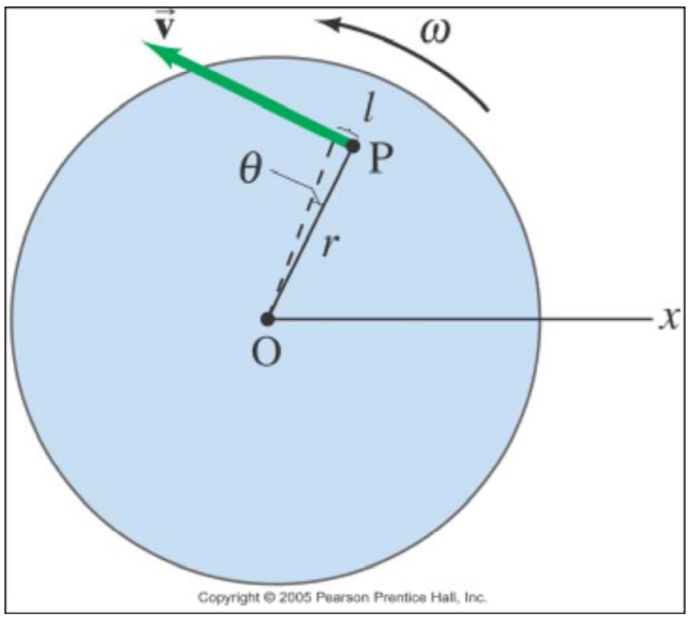
Since ω is the same for all points on a rotating object, α will also be the same for all points and angular acceleration will be expressed as rad/s^2.

A point traveling along a rotating object will have an angular velocity, as well as a linear velocity. If you remember from our unit on circular motion, an object traveling in a circular path has a velocity that is tangent to the circle. That is the same in this case when talking about linear velocity. Because it is tangent, we can solve for linear velocity by using the following equation:

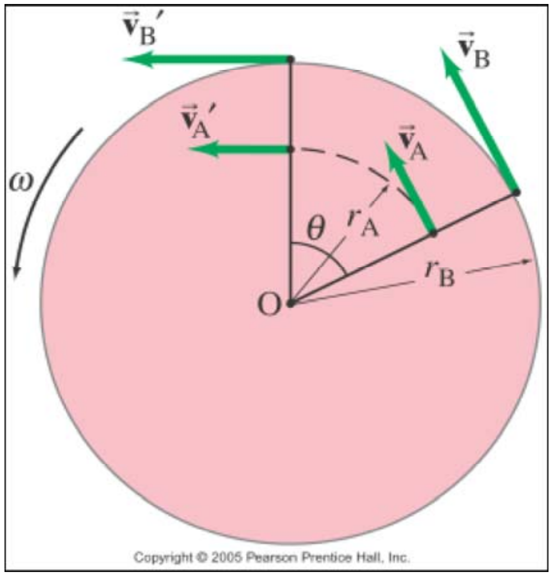
$$\boxed{v = r\omega}$$

By examining the equation, we can see that the further away from the axis of rotation, the greater the linear velocity will be. This is not true for angular velocity, remember that angular velocity is the same for all points traveling along the same rotating object.

The picture below shows how linear and angular velocities are related



The next picture shows how linear velocity increases as you move further away from the axis of rotation.



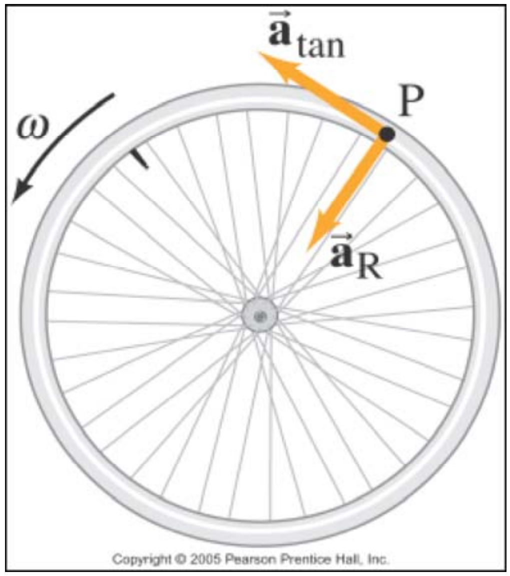
If a rotating object contains a linear velocity, it must also contain a linear acceleration. Linear acceleration however is a little more complicated than linear velocity. There is a linear acceleration vector that points tangent to the path of the circular motion just like linear velocity does. There is also a radial acceleration pointing towards the center of the circle (like we saw in our unit on circular motion). Radial acceleration is also called centripetal acceleration. So with these two types of acceleration, total linear acceleration is the vector sum of these two components.

$$\mathbf{a} = \mathbf{a}_{\tan} + \mathbf{a}_{R}$$

Where $\mathbf{a}_{\tan} = \mathbf{r}\alpha$ and $\mathbf{a}_{R} = \omega^2\mathbf{r}$

Centripetal (radial) acceleration depends on radius, therefore the further the point is from the axis of rotation, the greater centripetal acceleration it will feel.

The picture below shows the direction of the a_{tan} and a_R vectors



Example

1. A merry-go-round is initially at rest, after being pushed it obtains an angular acceleration of .060 rad/s^2 which increases its angular velocity for 8.0 seconds. At the end of the 8 seconds determine the

a. **Angular velocity of the merry-go-round**
b. **The linear velocity of a child located at 2.5 m from the center**
c. **The tangential (linear) acceleration of that child**
d. **The centripetal acceleration of the child**
e. **The total acceleration of the child**

GIVEN	SOLVING FOR
time = 8 s	**angular velocity**
anfular acceleration = .06 rad/s^2	

a. $\alpha = \Delta\omega/\Delta t$
$.06 = \omega/8$
$\boldsymbol{\omega = .48}$ **rad/s**

b. **GIVEN –** the radius of the child is 2.5 m
$v = r\omega$
$v = (2.5)(.48)$
$v = 1.2$ m/s

c. $a_{tan} = r\alpha$
$a_{tan} = (2.5)(.06)$
$a_{tan} = .15$ m/s^2

d. $a_R = \omega^2 r$
$a_R = (.48^2)(2.5)$

$a_R = .58\ m/s^2$

e. When finding the total acceleration, we know that it is a vector sum of a_{tan} and a_R and looking at the direction that these two vectors point along a circle, we can see that they are perpendicular to one another. Therefore in order to find the vector sum, we need to find the hypotenuse of the right triangle that they form. So…

$$a_{total}^2 = a_{tan}^2 + a_R^2$$
$$a_{total}^2 = .15^2 + .58^2$$
$$a_{total}^2 = .3599$$
$$a_{total} = \sqrt{.3599}$$
$$\mathbf{a_{total}^2 = .60\ m/s^2}$$

Frequency and Period

In our unit about circular motion we learned that **frequency** is equal to the number of revolutions per second. In this chapter we can relate angular velocity to frequency by thinking of it in terms of radians. One revolution corresponds to an angle or 2π radians.

Therefore 1 rev/s = 2π rad/s which will yield the equation:

$$\boxed{f = \frac{\omega}{2\pi}}$$

Don't forget the unit for frequency is the ***Hertz (Hz).***

The time required for one revolution is known as the **period** and is the inverse of frequency. So $T = 1/f$ because period is an amount of time, its units are the ***second***.

Example
1. A CD rotates at 6800 rpm (revolutions per minute) what is the angular velocity of the CD?

GIVEN
Frequency = 6800 / 60 = 113.3 Hz

SOLVING FOR
angular velocity

EQUATION

$$f = \frac{\omega}{2\pi}$$

113.3 = ω/6.28

ω = 711.7 rad/s

Constant Angular Acceleration

In unit 2, we learned the kinematics equations that we have used countless times throughout all units thereafter. These equations relate displacement, time, velocity and acceleration, assuming acceleration is constant. These same types of equations can be written in terms of angular displacement, time, angular velocity and angular acceleration assuming angular acceleration is constant. A main difference between them is that displacement (x) is replaced by radians (θ), v is replaced by ω, and a is replaced with α. Below is a chart that shows both linear kinematics equations alongside angular equations. You can see that they are very similar.

Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \frac{\omega + \omega_0}{2}$	$\overline{v} = \frac{v + v_0}{2}$

Example 1
1. A motor accelerates from rest to 10,000 rpm in 10 seconds, what is the average angular acceleration?

GIVEN
Frequency = 10,000/60 = 166.7 Hz
Time = 10 seconds
ω_o = 0 rad/s

SOLVING FOR
angular acceleration, α

EQUATIONS

$$f = \frac{\omega}{2\pi}$$

And

$$\omega = \omega_o + \alpha t$$

In order to find acceleration, we first need to solve for final angular velocity and then plug it in to find angular acceleration.

$$f = \frac{\omega}{2\pi} \quad 166.7 = \omega / 6.28 = 1046.7 \text{ rad/s}$$

$$\omega = \omega_o + \alpha t = \quad 1046.7 = 0 + \alpha(10)$$

$$\boldsymbol{\alpha = 104.7 \text{ rad/s}^2}$$

Example 2

A car engine slows from a frequency of 75 Hz to 20 Hz in 2.5 seconds. Find the acceleration of the car's engine.

GIVEN
Frequency initial = 75 Hz
Frequency final = 20 Hz
Time 2.5 s

SOLVING FOR
acceleration = ?

EQUATIONS
$f = \omega / 2\pi$
$\omega = \omega_o + \alpha t$

In order to solve for acceleration, we need to take the frequencies and find the initial and final angular velocities

$f = \omega / 2\pi$
$75 = \omega / 2\pi$
$\omega_o = 471.2$ rad/s

$f = \omega / 2\pi$
$20 = \omega / 2\pi$
$\omega = 125.7$ rad/s

now we can find acceleration

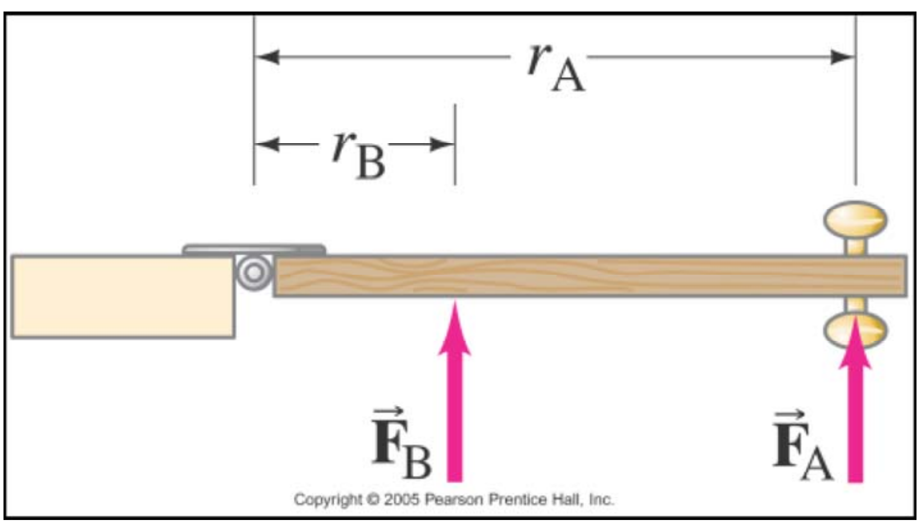
$\omega = \omega_o + \alpha t$
$125.7 = 471.2 + \alpha(2.5)$
$-345.5 = \alpha(2.5)$
$\boldsymbol{\alpha = -138.2}$ **rad/s^2**

Torque

Just like with linear motion, in order to get an object to move, a force is required. For rotational motion it is also very important where the force is applied as well as the direction of the force.

For example, think of using a wrench. The longer the wrench, the more force can be applied. The length at which the force is applied relative to the axis of rotation is called the ***lever arm***. When using something like a wrench, the further away from the axis of rotation, the less force is needed to move the object. In other words, not only does the amount of force matter, but the angular acceleration is also proportional to the perpendicular distance from the axis of rotation to the line along which the force acts (lever arm).

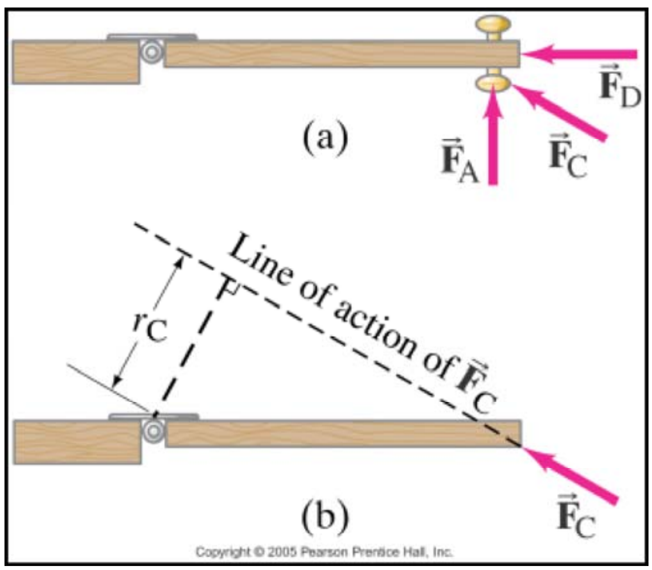
Below is a picture showing two forces acting at two different distances. This is a picture of a door on a hinge. And forces A and B are applied to open the door. At distance rB it will take more of a force to open the door compared to distance rA. Both forces are perpendicular to the lever arm.



The angular acceleration then must be equal to the product of the force and the lever arm. This product is known as ***torque.*** Torque is represented by the Greek lowercase letter tau (τ).

Depending on how a force is applied to create angular acceleration depends on the angle at which it is applied. If force is perpendicular to the lever arm, it will be the most effective. If it is at an angle, it will be slightly less effective.

Below is a picture of three different forces acting on a door in order to open it, F_A, F_C and F_D. F_A is acting perpendicular to the lever arm so it will need the least amount of force. F_C is acting at an angle. When a force acts at an angle, the easiest way to figure out exactly how force is actually used to cause an acceleration is to draw an imaginary line from the force vector to the axis of rotation. Then another line from the axis of rotation to the imaginary line in order to form a right angle. This is shown in picture B below. The distance then between the axis of rotation and the imaginary extension of the force vector is the actual lever arm distance being applied to the object in order to cause an angular acceleration. As you can see from the picture r_C is much shorter than where the force is actually being applied on the door. This shows how a force at an angle (F_C) is less effective than a force that is perpendicular. (F_A). Now looking at force F_D, it is somewhat common sense that the door would not move at all based on the location of the force.



Torque is defined as the perpendicular distance (lever arm) times the amount of force.

The units for torque is m*N. $\tau = r_{\perp} F$

So if the force is at an *angle,* the equation becomes $\boldsymbol{\tau = rF(\sin\theta)}$ where the θ is the angle between F and r. (HINT: you may have to draw a picture to see which angle to use, the angle the problem gives you, or its complimentary angle).

When more than one torque acts on an object, the angular acceleration is proportional to the sum of the torques. So if multiple torques are acting in the same direction, we can add the torque together to find acceleration. If a torque is acting in the opposite direction, we subtract the torques to find the net torque. Once again, torque can be negative. If the net torque is applied to accelerate the object in a counterclockwise direction, it is a positive torque. If the net torque is applied to accelerate the object in the clockwise direction, it is a negative torque.

Since torque is related to force and when a force or torque is applied to an object it causes an acceleration, it is easy to relate torque to Newton's Second Law of motion (F = ma).

The equation relating Newton's 2nd law and torque is written as $\tau = mr^2\alpha$

The quantity ***mr***^{2} in the equation above is known as the ***rotational inertia*** of a particle in motion. Since anything with mass has inertia and it also depends on its location, we can relate it to Newton's First Law in terms of inertia (the tendency for objects to resist change). Depending on the location of the mass m, an object can contain more than one rotational inertia, if these masses are along the same object. In this case, you would just find the rotational inertia for each mass and add them together. This then creates its own equation: Where *I* is rotational inertia. When there are many rotational inertias in a system, this is used to find the net torque.

$$I = \Sigma mr^2$$

Example 1
1. You push on a door perpendicular to the axis rotation with a force of 650 N at a distance of 35 centimeters from the door. How much torque is applied to opening the door?
GIVEN F = 650 N r = 35 cm SOLVING FOR Torque = ? EQUATION $\tau = r_\perp F$ τ = (.35)(650) = 227.5 m*N
Example 2
2. If the same door in example one has a mass of 3.2 kg, how fast would the door accelerate?

GIVEN
F = 650 N
r = 35 cm
$\tau = 227.5$ m*N
m = 3.2 kg

SOLVING FOR
acceleration = ?

EQUATION

$$\tau = mr^2\alpha$$

$227.5 = (3.2)(.35^2)\alpha$

$227.5 = .392(\alpha)$

$\boldsymbol{\alpha = 580.4 \text{ rad/s}^2}$

A look ahead:

In the next unit you will perform a laboratory investigation using a ladybug on a marry-go-round in order to observe and calculate angular velocity and acceleration of an object undergoing rotational motion.

Below are additional educational resources and activities for this unit.

[Unit 19 Resource 1]

[Unit 19 Resource 2]