

## **FURTHER INVESTIGATION INTO COMPLEX NUMBERS**

In a previous unit, we described the fundamental properties and definitions associated with Complex numbers. In addition, we practiced algebraic manipulations on these numbers, learned to find their location on the Complex Plane, and we assigned values to Complex numbers as vector magnitudes using the Pythagorean Theorem. In this unit, we will explore the geometric interpretation of Complex numbers in more depth. Particularly, we will interpret the four arithmetic operations of addition, subtraction, multiplication, and division for Complex numbers geometrically on the Complex plane. Later in the course, we will examine alternate methods for graphing numbers and functions in general, and how these alternate methods can help us interpret Complex numbers and functions in even greater depth.

The Geometry of Complex Addition and Subtraction

The Geometry of Complex Multiplication

Using the Tangent to Determine the Angular Rotation for Complex Multiplication

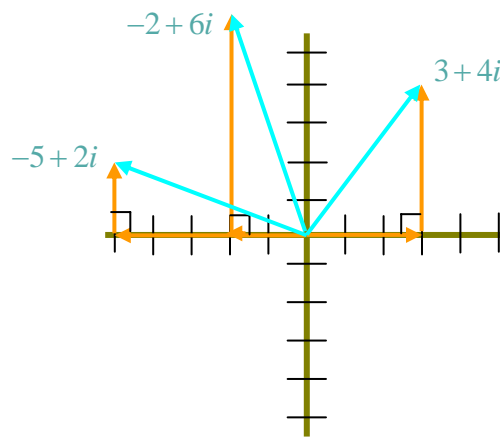
Using the Graphing Calculator to find the Rotation Angle

## The Geometry of Complex Addition and Subtraction

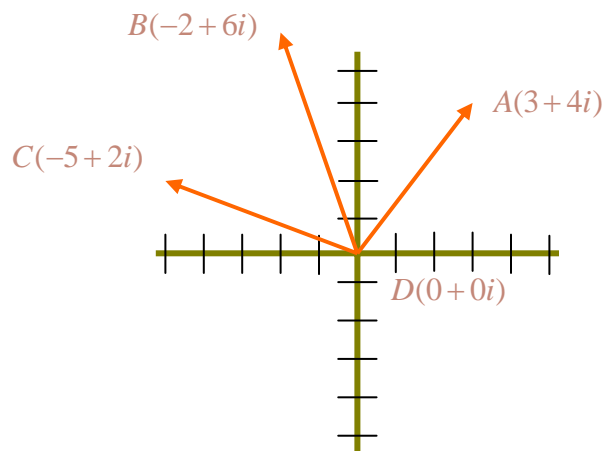
Recall that in order to add or subtract any two Complex numbers, we simply match the real and Complex components of each number as if they were algebraic like terms.

*Example #1:* Add:  $(3+4i) + (-5+2i) = (3-5) + (4+2)i = -2+6i$

We also know from a previous unit that a vector magnitude for each of the above numbers can be calculated using the Pythagorean Theorem, and that this vector, graphed as a ray from the origin, indicates the unique location of every Complex number on the Complex Plane.



Let us examine this graph as vector quantities only without the right triangle components illustrated and label each point as A, B, C and D as indicated.



Recall again from a previous unit that graphing Complex numbers is virtually identical to graphing points on the real  $xy$ -plane. In this respect, we can assign the following coordinates to the complex numbers shown above:

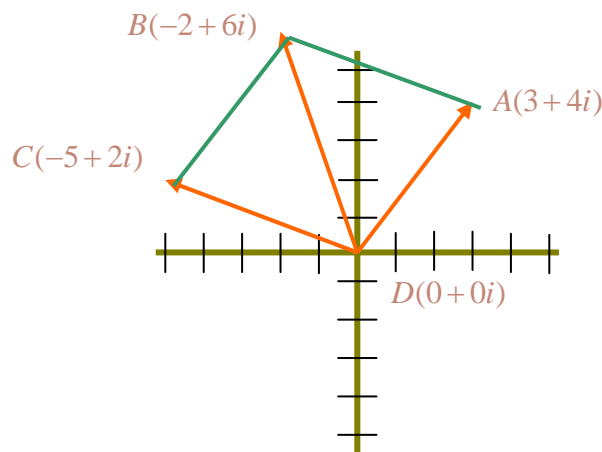
$$A(3,4) \quad B(-2,6) \quad C(-5,2) \quad D(0,0)$$

We now use our linear equations to calculate the slopes of the lines between any two consecutive points.

$$m(AB) = \frac{6-4}{-2-3} = -\frac{2}{5} \quad m(BC) = \frac{2-6}{-5-(-2)} = \frac{4}{3}$$

$$m(CD) = \frac{0-2}{0-(-5)} = -\frac{2}{5} \quad m(AD) = \frac{4-0}{3-0} = \frac{4}{3}$$

Notice the slopes for lines between two sets of points on opposite sides of the diagram are identical. If  $\overline{AB}$  and  $\overline{BC}$  are connected in the diagram, quadrilateral  $ABCD$  is formed.



From geometry this quadrilateral is a **parallelogram** and  $\overline{DB}$  is a diagonal of this parallelogram. Therefore, the geometric interpretation for Complex addition is as follows:

### Complex Addition

Let  $a+bi, c+di \in \mathbb{C}$  with vector representation,  $\vec{v} = a+bi$ ,  $\vec{w} = c+di$ . Further, we allow that  $\vec{v}$  and  $\vec{w}$  are represented on the Complex Plane by the points:  $\vec{v}(AB)$ ,  $\vec{w}(AD)$ , where  $A = 0+0i$ .

By definition:  $\vec{v}(AB) + \vec{w}(AD) = (a+bi) + (c+di) = (a+c) + (b+d)i = \vec{s} \in \mathbb{C}$

Allow that  $\vec{s}$  be represented on the complex plane as  $\vec{s}(AC)$ .

Then  $ABCD$  is a parallelogram on the Complex Plane with diagonal  $\overline{AC} = \vec{s}$  and  $\vec{v} + \vec{w} = \vec{s}$  defines vector addition.

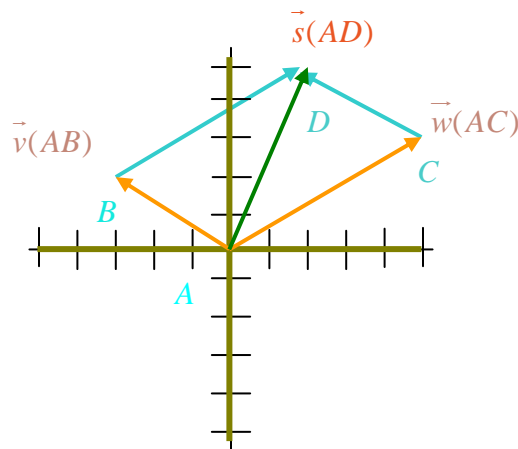
From the previous example, the parallelogram and diagonal  $\overline{DB}$  represents the addition of  $(3 + 4i) + (-5 + 2i) = -2 + 6i$ .

The parallelogram  $DABC$  is called the “characteristic parallelogram” of vector addition and has many unique and valuable properties in science, economics, and business. Although most of these properties will not be discussed in this course, one is worth noting. Each side of the parallelogram can be seen to represent a “path” or “route” to point B starting at point D. One path to B is D to C to B. A second path to B is to travel D to A to B. Because  $DABC$  is a parallelogram, the total distance to B from D along either path is equal. In a real-world situation, this implies that alternate paths or procedures can be found, both of which provide the same result. In this way, a blocked route or faulty procedure can be discarded and replaced with a new process without a significant loss in time, effort, or monetary expenditure.

*Example #2:* Graph the Characteristic Parallelogram for:  $\vec{v} + \vec{w} = \vec{s}$ ,  
where:  $\vec{v} = (-3 + 2i)$ ;  $\vec{w} = (5 + 3i)$

*Step #1:* Add:  $\vec{v} + \vec{w} = (-3 + 2i) + (5 + 3i) = (2 + 5i) = \vec{s}$

*Step #2:* Graph:  $\vec{v}$ ,  $\vec{w}$ ,  $\vec{s}$  and connect  $\overline{CD}$ ;  $\overline{BD}$ , to form parallelogram  $ABCD$ .



## The Geometry of Complex Multiplication

Complex multiplication depends on “[Property of Recursion](#)” and how this affects the final [products](#)’ location on the Complex Plane:

**Recursion Value**

**Recursion Value as a Complex Number**

$$i^0 = 1$$

$$1 + 0i$$

$$i^1 = i$$

$$0 + i$$

$$i^2 = -1$$

$$-1 + 0i$$

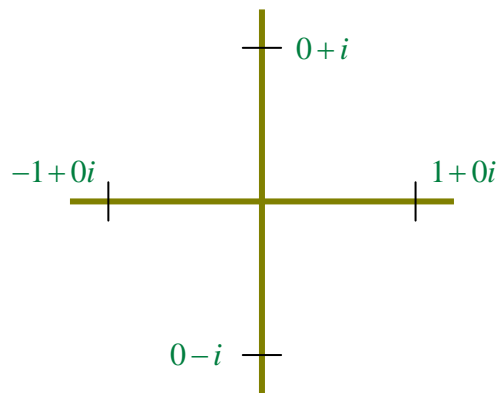
$$i^3 = -i$$

$$0 - i$$

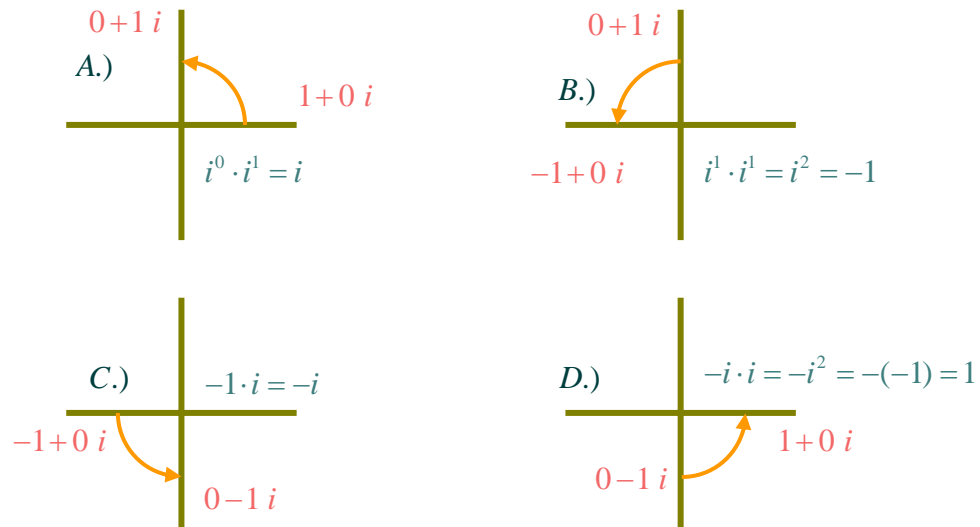
$$i^4 = 1$$

$$1 + 0i$$

Begin by graphing each recursion value on the Complex Plane as a Complex number.



Beginning with  $i^0 = 1$ , each step in the recursion process is analogous to multiplying the current value by  $i^1$ . With each successive multiplication, the new Complex number assumes a new location on the Complex Plane that is exactly  $90^\circ$  counterclockwise from its previous position, and on either the  $x$  or  $y$  axis. Represented graphically, each multiplication by  $i$  results in the following:



The above graph should be read in order  $A \rightarrow B$

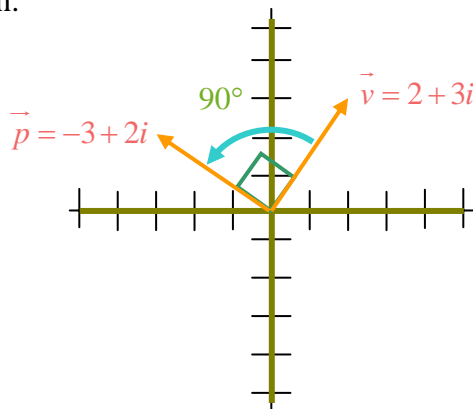
### Description for Complex Multiplication of "1i"

In the Complex Plane, every multiplication by  $i$  on a Complex number has the same effect as rotating the Complex number counterclockwise by an angle of  $90^\circ$ .

*Example #1:* Multiply:  $\vec{v} = 2 + 3i$  times  $i$  and graph  $\vec{v}$  and the product  $i \cdot \vec{v} = \vec{p}$  on the Complex plane.

*Step #1:* Multiply:  $i(2 + 3i) = 2i + 3i^2 = -3 + 2i = \vec{p}$

*Step #2:* Graph:



## Complex Multiplication by a value of "bi"

In the previous example, a rotation of  $90^\circ$  occurs when a Complex number is multiplied by "1 i". However, just as with the real numbers, when multiplying by "1", the **magnitude** of the product remained the same:

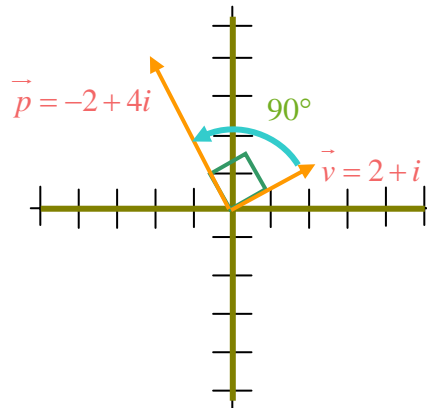
$$\begin{aligned} |\vec{v}| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\ |\vec{p}| &= \sqrt{3^2 + 2^2} = \sqrt{13} \end{aligned}$$

The geometric result of multiplying a Complex number by the multiple,  $bi$ , DOES affect the magnitude.

*Example #2:* Multiply:  $\vec{v} = 2 + i$  by  $2i$ . Graph  $\vec{v}$ ,  $2i \cdot \vec{v} = \vec{p}$ , and find  $|\vec{v}|$ , and  $|\vec{p}|$ , the vector magnitudes.

*Step #1:* Multiply:  $2i(2 + i) = 4i + 2i^2 = -2 + 4i$

*Step #2:* Graph:



*Step #3:* Find:  $|\vec{v}|$  and  $|\vec{p}|$

$$\begin{aligned} |\vec{v}| &= \sqrt{2^2 + 1^2} = \sqrt{5} & |\vec{p}| &= \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Multiplying a Complex number by a multiple of  $bi$  affects the magnitude of the Complex number by a factor of  $b$ . Again, this is similar to multiplication of real numbers by any number equal to 1. The difference for Complex numbers is that the change in magnitude is accompanied by a  $90^\circ$  change in position.

## Complex Multiplication by a value of " $a + bi$ "

We finally arrive at a geometric interpretation for full Complex multiplication. As we have seen, Complex multiplication by  $i$  or  $bi$  has two affects on the original Complex number:

- 1.) The magnitude of the number is increased or decreased by the factor " $b$ ".
- 2.) The position of the Complex number is affected by a  $90^\circ$  counterclockwise rotation through the Complex Plane.

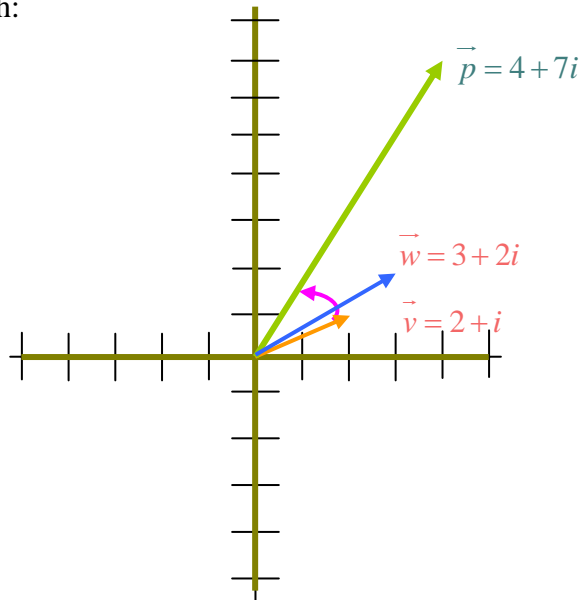
Since multiplication of a Complex number by  $a + bi$  is simply an extension of our previous descriptions, both the magnitude and rotation of the original number will be affected. However, although the effect on the magnitude may be somewhat predictable, the amount of rotation for our new number is a bit unexpected.

*Example #3:*

- 1.) Multiply  $\vec{v} = 2 + i$  by  $\vec{w} = 3 + 2i$ .
- 2.) Graph  $\vec{v}$ ,  $\vec{w}$ , &  $\vec{p}$  (the product vector).
- 3.) Determine the magnitude of each.

*Step #1:* Find:  $\vec{v} \cdot \vec{w} = (2 + i)(3 + 2i) = 6 + 4i + 3i + 2i^2 = 6 - 2 + 7i = 4 + 7i = \vec{p}$

*Step #2:* Graph:

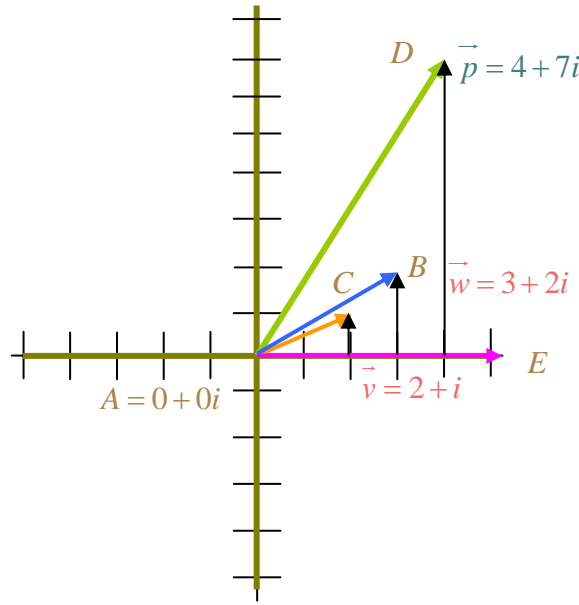


As is evident in this graph, the rotation was not by a multiple of  $90^\circ$ . The exact amount of rotation can be determined precisely using methods from **Trigonometry**, a topic that will be covered in detail later in the course. In this unit, we will determine the rotation amount as a preview to our study of Trigonometry.



## Using the Tangent to Determine the Angular Rotation for Complex Multiplication

When two Complex numbers are multiplied, the angle of rotation of the product vector can be identified as the acute angle in the right triangle that defines the components of the product vector on the Complex plane. From the previous example



The legs and magnitude of the characteristic triangle for each vector are as follows:

$$\begin{aligned} \vec{v}: \text{legs } a = 2, b = 1 & \quad |\vec{v}| = \sqrt{5} \\ \vec{w}: \text{legs } a = 3, b = 2 & \quad \Rightarrow \quad |\vec{w}| = \sqrt{13} \\ \vec{p}: \text{legs } a = 4, b = 7 & \quad |\vec{p}| = \sqrt{65} \end{aligned}$$

The amount of rotation for each vector from the  $x$ -axis is as follows:

$$\begin{aligned} \vec{v} &\Rightarrow m\angle EAC \\ \vec{w} &\Rightarrow m\angle EAB \\ \vec{p} &\Rightarrow m\angle EAD \end{aligned}$$

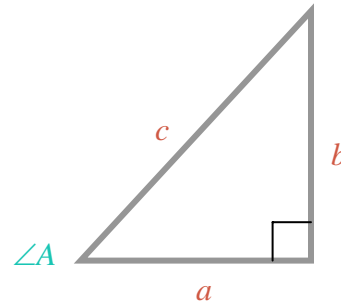
Finding the measure of each angle, we will determine the amount of rotation that occurred under vector multiplication.

In Trigonometry there is a way to find angle measures in right triangles if two sides of the triangle are known. By making ratios out of the lengths of the sides of the right triangle, it can be shown that each ratio (reduced to the lowest terms) is uniquely associated with an angle of given measurement. In total, there are six ratios that can be made from the sides of a right triangle. Three of these six are listed briefly below and will be discussed in more detail in later units:

$$\sin \angle A = \frac{b}{c} : \text{ sine of } \angle A$$

$$\cos \angle A = \frac{a}{c} : \text{ cosine of } \angle A$$

$$\tan \angle A = \frac{b}{a} : \text{ tangent of } \angle A$$



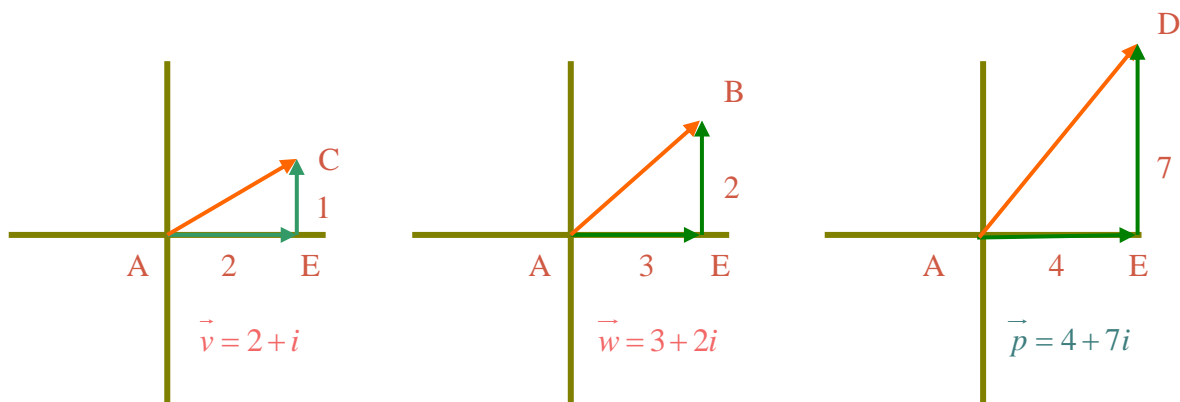
From this list, the ratio of the  $\tan \angle A$  is pertinent to this unit.

In the previous example, the characteristic triangles for  $\vec{v}$ ,  $\vec{w}$  and  $\vec{p}$  were found and the legs of each triangle are known.

- a.) For  $\vec{v}$ ;  $a = 2, b = 1$
- b.) For  $\vec{w}$ ;  $a = 3, b = 2$
- c.) For  $\vec{p}$ ;  $a = 4, b = 7$

From the list of trigonometric ratios presented above the ratio for  $\frac{b}{a}$  of these sides is defined to be the tangent of the angle  $A$ , whose vertex is at the origin,  $0 + 0i$ .

Graphed separately for clarity:



Since we can identify the ratio of the sides of each triangle as the tangent of an unknown angle, our task is to use this ratio to identify the measure of each angle. Fortunately, the graphing calculator is equipped to find these measurements.

## Using the Graphing Calculator to find the Rotation Angle

If two sides of a right triangle are known, then finding the measure of the angle associated with the ratio of these sides is called finding the “arctangent” of the angle. In math, we denote the arctangent as  $\tan^{-1} \angle A$ . (Note: the “-1” is not an exponent. As with functions, this indicates a form of inverse of the tangent.) On your calculator, this feature is found by pressing  $\boxed{2\text{nd}}$ ,  $\boxed{\text{TAN}}$ .

Procedure to find the arctangent:  $(\tan^{-1} \angle A)$ :

- 1.) Press,  $\boxed{\text{MODE}}$ , highlight ‘Degree’ and press ‘ $\boxed{\text{ENTER}}$ ’ to calculate the angle measure in degrees. Press  $\boxed{2\text{nd}}$ ,  $\boxed{\text{MODE}}$  to return to the home screen.
- 2.) Press  $\boxed{2\text{nd}}$ ,  $\boxed{\text{TAN}}$  and enter the ratio ‘1/2’ for the first vector,  $\vec{v}$ . Press  $\boxed{)}$ ,  $\boxed{\text{ENTER}}$  and the initial angle measure is reported as  $26.57^\circ$ .
- 3.) Press  $\boxed{2\text{nd}}$ ,  $\boxed{\text{TAN}}$  and enter the ratio ‘2/3’ for the vector  $\vec{w}$ . Press  $\boxed{)}$ ,  $\boxed{\text{ENTER}}$  and the angle measure is  $33.69^\circ$ .
- 4.) Press  $\boxed{2\text{nd}}$ ,  $\boxed{\text{TAN}}$  and enter the ratio ‘7/4’ for the vector  $\vec{p}$ . Press  $\boxed{)}$ ,  $\boxed{\text{ENTER}}$  and the angle measure is  $60.26^\circ$ .

The obvious conclusion for this example is that the rotation angle for the product vector,  $\vec{p}$ , is the sum of the rotation angles for  $\vec{v}$  and  $\vec{w}$ .

$$\tan_1^{-1}(\vec{v}) + \tan_2^{-1}(\vec{w}) = \tan_3^{-1}(\vec{p}) \Rightarrow \tan_1^{-1}\left(\frac{1}{2}\right) + \tan_2^{-1}\left(\frac{2}{3}\right) = \tan_3^{-1}\left(\frac{7}{4}\right) \Rightarrow 26.57^\circ + 33.69^\circ = 60.26^\circ$$

This is indeed the case and is stated in the following theorem.

### Theorem: Vector Multiplication

Let  $\vec{v}$  &  $\vec{w} \in \mathbb{C}$ , be complex vectors and let  $\vec{p}$  be the product vector of  $\vec{v} \cdot \vec{w} = \vec{p}$ . If  $\tan_1^{-1}(\vec{v})$  and  $\tan_2^{-1}(\vec{w})$  are the rotation angles of  $\vec{v}$  &  $\vec{w}$ , respectively, then:

$$\tan_1^{-1}(\vec{v}) + \tan_2^{-1}(\vec{w}) = \tan_3^{-1}(\vec{p})$$

Note the following:

- 1.) When using this method, be sure to always enter the ratio of the sides of the triangle as the vertical component divided by the horizontal component ( $b/a$ ).
- 2.) The actual rotation angle for the product of two vectors may actually be  $> 360^\circ$ . Vector multiplication may rotate the angle through one or many complete turns of a circle. For the purposes of this course, we will assume a simple rotation of less than  $360^\circ$ .

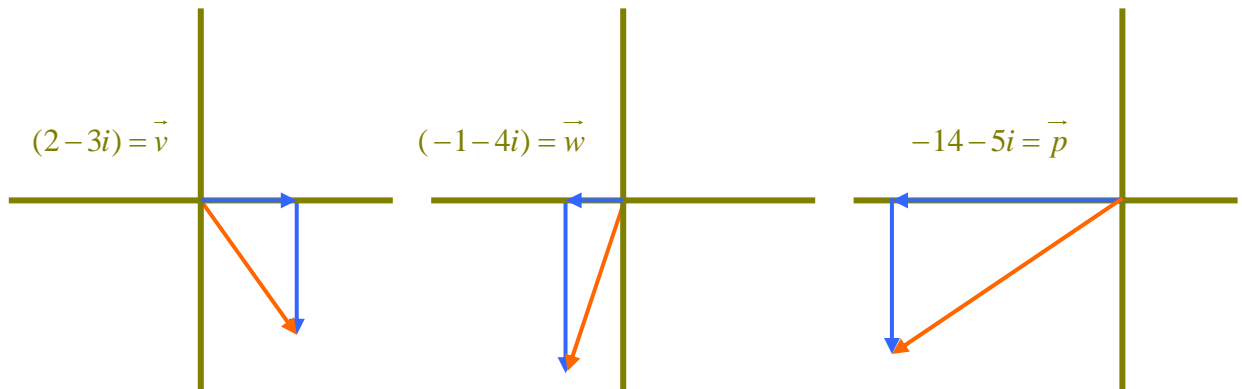
*Example #1:*

- 1.) Find the product vector  $\vec{v} = 2 - 3i$ ;  $\vec{w} = -1 - 4i$ .
- 2.) Graph  $\vec{v}$ ,  $\vec{w}$ , &  $\vec{p}$
- 3.) Find the rotation angles for  $\vec{v}$ ,  $\vec{w}$ , &  $\vec{p}$  using the arctangent ( $\tan^{-1}$ ).

*Step #1:* Find  $\vec{p}$ :

$$(2 - 3i)(-1 - 4i) = -14 - 5i = \vec{p}$$

*Step #2:* Graph: (Graphed separately for clarity; graph not shown to scale)



*Step #3:* When finding the arctangent for vectors in Quadrant #2, #3, or #4, we must also consider the sign of the values associated with each leg of the right triangle. This indicates not only amount, but also direction of the rotation angle, a topic to be covered in a later unit.

- a.) For  $\vec{v}$ ;  $\tan^{-1}(-3/2) = -56.31^\circ$
- b.) For  $\vec{w}$ ;  $\tan^{-1}(-4/-1) = 75.96^\circ$
- c.) For  $\vec{p}$ ;  $\tan^{-1}(-5/-14) = 19.65^\circ$