THE NATURE OF COMPLEX NUMBERS

In a previous unit, we learned that numbers are classified according to common characteristics and type. In that unit we grouped numbers into one of eight categories and located those categories on a “Number Tree”. At the top of this tree are the Complex Numbers. Using our definitions of sets and subsets, we described every number on the number tree as a subset of the Complex Numbers. In essence, all numbers are complex. In this unit we will examine the basic features of Complex Numbers and their mathematical properties. We will perform algebraic operations on Complex Numbers and we will introduce the Complex Number Plane in order to graph complex numbers and study their magnitude.

Basic Operations on Complex Numbers

Recall our definition of Complex Numbers:

Complex Numbers: Let, $a,\ b \in \mathbb{R}$ and $i = \sqrt{-1}$, then a number of the form:

$a \pm bi$, is called a Complex Number.

In order to begin a survey of complex numbers, we will look at the imaginary component, $\sqrt{-1}$, and what it represents to this special class of numbers.

Recursion

From the above definition: $i = \pm\sqrt{-1}$

This relationship is called the "Fundamental Form" of an imaginary number. Applying the laws of exponents and radicals to this relationship results in the following "identity" for imaginary numbers:

$$i = \pm\sqrt{-1} \quad \Rightarrow \quad i^2 = -1$$

Squaring this identity again:

$$(i^2) = i^4 = (-1)^2 = 1$$

Recall from the laws of exponents that $i^0 = 1$; therefore, it can be concluded that $i^4 = i^0 = 1$.

Through further application of the exponent laws, the following table of values for increasing powers of i^n can be constructed.

Recursion Table for i

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

The table demonstrates that powers of imaginary numbers can take on only one of four values:

$$1, \quad i, \quad -1, \quad -i$$

All higher powers of ‘ i ’ will ‘recur’ to one of these four values.

Example #1: Establish the recursion value for the following.

a.) i^8 b.) i^{34} c.) i^{2355}

From the recursion table, every $i^4 = i^0 = 1$. Using this fact, the above problems are analyzed as:

a.) $i^8 = (i^4)^2 = 1^2 = 1$

b.) $i^{34} = i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^4 \cdot i^2$
$= 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot i^2 = 1 \cdot (-1) = -1$

c.) However, for, i^{2355}, listing every i^4 is impractical. Instead, we realize that determining the repetition of i^4 is the same as determining how many times 4 divides into 2,355. The remainder from this division is the last exponent power on i and can be found in the recursion table to determine this formal value:

$$\begin{array}{r}
588 \\
4\overline{)2355} \\
\underline{20} \\
355 \\
\underline{32} \\
35 \\
\underline{32} \\
3
\end{array}$$

Therefore $i^{2355} = (i^4)^{588} \cdot i^3 = 1 \cdot i^3 = i^3 = -i$

Now that the recurrence for imaginary numbers is established, we can examine how these values influence the value of complex numbers in general.

The Algebra of Complex Numbers

The manipulation of Complex Numbers according to the laws of arithmetic involves many of the standard rules with some modifications.

Let, $a+bi,\ \ c+di \in \mathbb{C}$, then

1.) $(a+bi)\pm(c+di)=(a\pm c)+(b\pm d)i$

2.) $(a+bi)\cdot(c+di)=ac+adi+bci+bdi^2=ac+bd(-1)+adi+bci=(ac-bd)+(ad+bc)i$

3.) $\dfrac{a+bi}{c+di}=\dfrac{(a+bi)}{(c+di)}\cdot\dfrac{(c-di)}{(c-di)}=\dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

*This last property for division is also called the "property of complex conjugates". Recall that we encountered the conjugate of a radical expression in an earlier unit.

4.) $c(a+bi)=ac+bci \quad : \quad di(a+bi)=adi+bdi^2=-bd+adi$

Example #1: Perform the indicated operations for each pair of complex numbers.

a.) $(3-6i)-(7+4i)=(3-7)+(-6i-4i)=-4-10i$

b.) $5(15+i)=75+5i$

c.) $5i(15+i)=75i+5i^2=75i+5(-1)=-5+75i$

d.) $(6+5i)(3-2i)=18-12i+15i-10i^2=18-10(-1)+3i=28+3i$

e.) $\dfrac{2}{3-7i}=\dfrac{2}{(3-7i)}\cdot\dfrac{(3+7i)}{(3+7i)}=\dfrac{6+14i}{9+21i-21i-49i^2}=\dfrac{6+14i}{9-49(-1)}=\dfrac{6+14i}{9+49}=\dfrac{6+14i}{58}$

*This last example can and should be reduced further to:

$$\frac{2(3+7i)}{58}=\frac{3+7i}{29}$$

Complex Numbers as Solutions to Quadratic Equations

Recall from a previous unit that according to the Fundamental Theorem of Algebra (FTA), every polynomial of degree 'n' has exactly 'n' complex solutions.

Consider the equation: $x^2 + x + 1 = 0$

This equation is non-factorable over the real numbers. However, the equation can be placed in the quadratic formula, where:

$$a = 1, \quad b = 1, \quad c = 1$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

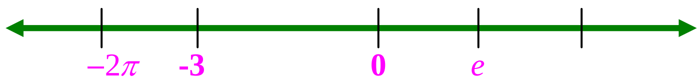
Example #1: Use the Quadratic formula to solve:

$$x^2 + 2x + 17 = 0: \qquad a = 1, \quad b = 2, \quad c = 17$$

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 17}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 - 68}}{2} = \frac{-2 \pm \sqrt{-64}}{2} = \frac{-2 \pm \sqrt{-1 \cdot 64}}{2} = \frac{-2 \pm 8i}{2} = -1 \pm 4i$$

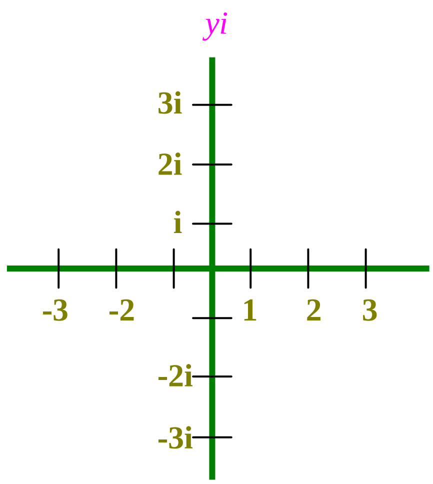
Graphing Complex Numbers (The Complex Plane)

As we know, any real number can be located on a number line:



However, since the number line finds the position of Real numbers, the question arises as to where do we locate Complex numbers?

Because the Complex numbers incorporate both real and imaginary components, their position on a graph must account for these components. The result is that Complex x numbers are found on the Complex Plane. Below is an example of this plane:



We notice immediately the similarities between this diagram and the usual set of coordinate axes we use to graph points, equations and functions.

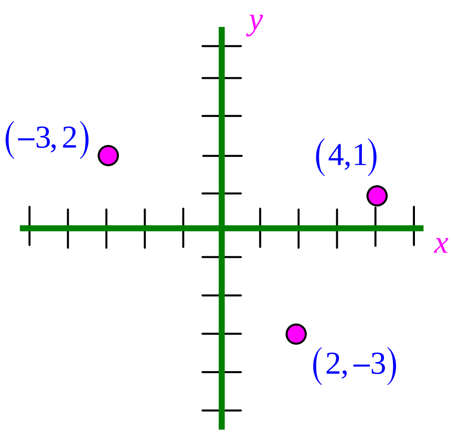
When we consider a standard point, such as $(2,-3)$, we see that when graphing, we must consider two distinct components:

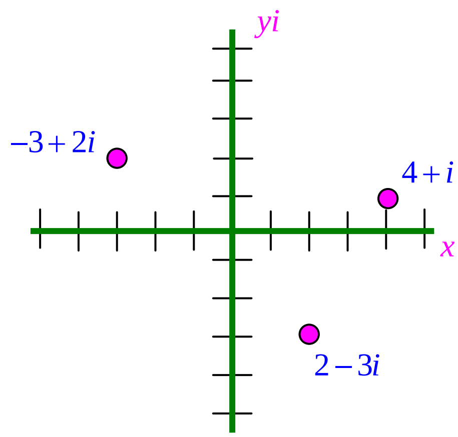
1.) The x component: 2
2.) The y component: -3

For a Complex number, such as $2-3i$, we also consider two components:

1.) The x component: 2
2.) The yi component: $-3i$

Although the Complex number, $2-3i$, is taken to be one number, whereas the point $(2,-3)$ are two distinct real numbers in a relation, we can graph complex numbers in exactly the same manner that we graph points.

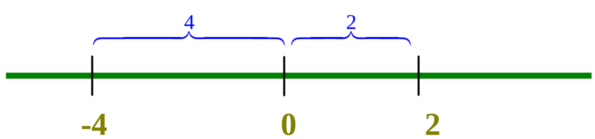




As we see on these graphs, the location of each Complex number on the Complex Plane is analogous to graphing a standard point, with the same real components, on the traditional coordinate axes. The distinction is that each Complex number is a single value, but each standard point is two values under a relation.

We will next examine what is meant when we speak of the "value" of a Complex number.

To speak of a Real number's "value" is to examine the number's distance, from zero, on a number line.

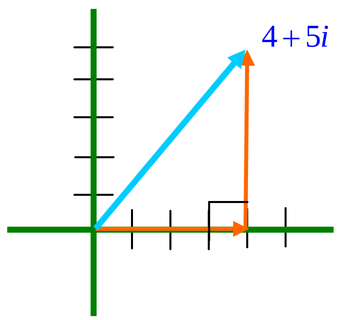


This is to say that the value of a Real number is that number's "absolute value" without regard to its direction from zero.

A similar description for the "value" of Complex numbers can also be made. On the real number line, we can "count" the number of units a number is from zero. On the Complex Plane we must find the number's distance from the origin, $0+0i$. To accomplish this task, a geometric interpretation of a Complex number will be explained in the next topic.

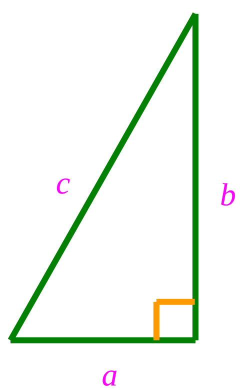
The Magnitude of a Complex Number

To graph any point on the coordinate axes or any Complex number on the Complex Plane, we follow the same procedure as graphing a standard point with Real number values. First we locate the value of our number in the horizontal, or x, direction, and then we turn 90 degrees and proceed to find our number's vertical value in the y direction. For both a point and a Complex number this process traces the "legs" of a right triangle.



The length of the third side of this triangle, the **hypotenuse**, is the complex number's distance from the origin on the complex plane. This distance, called the "magnitude" of the complex number, can be found using the **Pythagorean Theorem** for right triangles from Geometry.

Recall the **Pythagorean Theorem:** In a right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse.

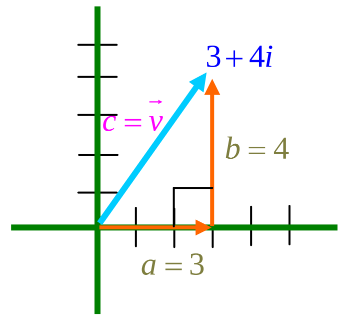


If a & b are legs, and c is the hypotenuse, then

$$a^2 + b^2 = c^2$$

Example # 1: Find the magnitude of the complex number $3+4i$.

The diagram for this problem can labeled as in the right.



The information from this diagram is then placed in the Pythagorean Theorem to obtain:

$$3^2+4^2=c^2$$
$$25=c^2$$
$$5=c=\vec{v}$$

Two aspects of the magnitude of a Complex Number are worth noting:

1.) For complex numbers, the hypotenuse of the right triangle is called a "vector"(labeled, $\vec{v}$) and its length is called the "vector magnitude".

2.) The vector (hypotenuse) and legs of the right triangle associated with a complex number are labeled as rays and not line segments. This identifies the fact that a complex number's position on the Complex Plane is found in a certain direction from $0+0i$.

The "Equality" of Complex Numbers

The Complex Numbers are unique in mathematics not only because their "number line" is really a plane, but also because they seem to defy one of the most fundamental of all properties for any number; uniqueness.

The unique value of any Real number is stated in the following property:

Trichotomy: Let $a, b \in \mathbb{R}$, then only one of the following is true:

$$a > b, \quad \text{or} \quad a = b, \quad \text{or} \quad a < b$$

This property, also known as the "Well Ordering Principle", simply states that no two real numbers can be both different and equal at the same time or; two distinct real numbers cannot both have the same magnitude. Aside from the fact that additive inverses both have the same absolute value, and therefore the same magnitude in the respect that distance must always be positive, the two real numbers, a & $-a$, can always be arranged as, $-a < a$, on the real number line. However, for the two complex numbers, $-a + bi$ & $a + (-b)i,$ we cannot say which number is larger or smaller, which is labeled first (since one is in Quadrant II and the other in Quadrant IV); and yet, both numbers have the same magnitude if evaluated using the Pythagorean theorem.

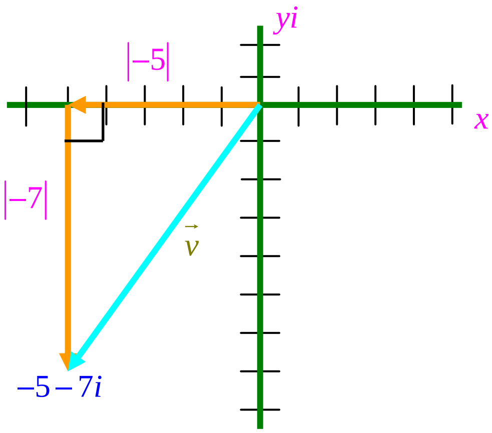
Another way to examine this is to note that i may be written as $0 + 1i$ while $0 = 0 + 0i$. Therefore, by definition, $0 < i$. Obviously, $-1 < 1$. Multiplying by i we get $-i < i$. Multiplying by i one more time produces: $-i^2 < i^2 \quad \Rightarrow \quad -(-1) < -1 \quad \Rightarrow \quad 1 < -1$. This is obviously false. If i behaved, as do real numbers in this situation, then repeated multiplication of both sides of the inequality by a positive number would not alter the direction of the inequality.

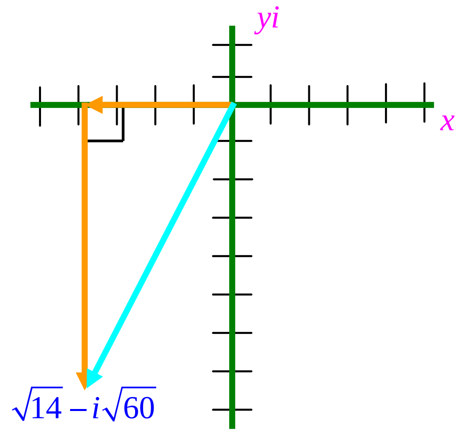
The location of any two real numbers are unique on the real number line and can only be arrived at by one and only one method; go to the right of zero, x number of units, and you will always locate the real number "h". Complex numbers do not appear to posses this same quality. Consider the two Complex numbers:

$$-5 - 7i \quad \& \quad -\sqrt{14} - 2i\sqrt{15}$$

For any comparison of the real components of the two numbers, the numbers are not equal. $-5 \neq -7 \neq -\sqrt{14} \neq -2\sqrt{15}$.

In addition, both numbers are located in quadrant III, which does not indicate an order between the numbers as well. Yet, for the purposes of the Complex Plane, the two numbers are equal. This can be seen when we locate both numbers on the complex plane and use the Pythagorean Theorem to assess their value.





$\sqrt{14} \approx 3.75 \qquad \sqrt{60} \approx 7.75$

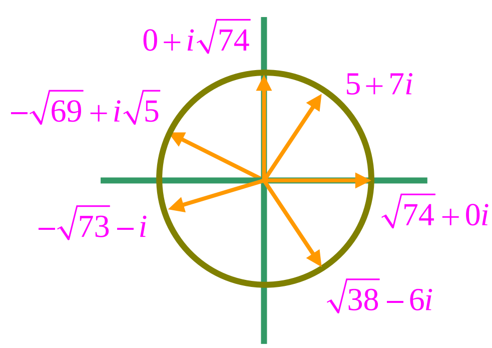
$$5^2 + 7^2 = |\vec{v}_1|^2$$
$$25 + 49 = |\vec{v}_1|^2$$
$$74 = |\vec{v}_1|^2$$
$$\sqrt{74} = \vec{v}_1$$

$$(\sqrt{14})^2 + (\sqrt{60})^2 = |\vec{v}_2|^2$$
$$14 + 60 = |\vec{v}_2|^2$$
$$74 = |\vec{v}_2|^2$$
$$\sqrt{74} = \vec{v}_2$$

In fact, there are an infinite number of Complex Numbers equal to the above two numbers such as:

$$\sqrt{74} + 0i, \quad 0 - i\sqrt{74}, \quad -3 + i\sqrt{65}, \quad \sqrt{38} - 6i, \quad 5 + 7i, \quad 0 + i\sqrt{74}$$

If all these numbers of equivalent magnitude are graphed on the same plane, they would trace a circle whose radii are equal $(r = \sqrt{74})$.



We will return to the circular nature of a complex number later in the course.