RADICAL EQUATIONS AND EXPRESSIONS

As was noted in previous units, non-solutions to an equation can be as important to a problem (and sometimes more important) as is a solution. In rational expressions and equations, values that caused division by zero had to be identified and restricted from the problem. Also recall that in addition to these “natural” restrictions, we may select our own restrictions for a problem as was the case with piecewise functions. For this unit, we will examine restrictions that may be placed on a problem which may be due to our choice, or because of the laws of mathematics, or as a combination of the two. First we will review methods for solving equations that involve radical expressions and simplify rational expressions with radical denominators.

Simplifying Radical Expressions

Recall that an expression of the form

$$\sqrt[a]{x^n}$$

is called a radical expression. The parts of this expression are:

1.) " a " – The 'index', or the "root" of the expression.
2.) " $\sqrt[a]{}$ " – The 'radical' symbol.
3.) " x^n " – The radicand.

This expression can be rewritten as a fractional power on ' x ' of the form:

$$x^{n/a}\text{, where } a,\ n \in \mathbb{Z}$$

Either expression is not considered to be written in simplest form unless $n \le a$. Therefore, when we encounter the following example, we use our laws of radicals (Law #1) to simplify the expression.

Example #1: Simplify: $\sqrt[3]{x^4}$

Answer: $\sqrt[3]{x^4} = \sqrt[3]{x^3 \cdot x} = \sqrt[3]{x^3} \cdot \sqrt[3]{x} = x\sqrt[3]{x}$

*Note: all steps are demonstrated for emphasis. Students may combine multiple steps.

In addition to the above example, a radical expression is not considered to be in simplest form if the expression contains a radical in the denominator of a rational expression such as:

Example #2: Simplify: $\dfrac{5}{\sqrt{x}}$

For these expressions we seek to change the "appearance" of the expression without changing its value. In order to accomplish this task, we make use of the special multiplicative value of the number one.

$$1 \cdot x = x \cdot 1 = x$$

The above identity verifies that the number one will not change the value of an expression when '1' multiplies the expression. Using this property and our laws of radicals, we can simplify the above expression in the following manner:

$$\frac{5}{\sqrt{x}} = \frac{5}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{5\sqrt{x}}{\sqrt{x^2}} = \left|\frac{5\sqrt{x}}{x}\right|$$

(The answer is placed in absolute value symbols to account for $\pm$ answers when finding a '$\sqrt{\ }$').

By selecting an expression whose value equals 1, $\frac{\sqrt{x}}{\sqrt{x}}$, we have changed the appearance of our expression, but not its value.

Example #3: Simplify: $\frac{3}{4+\sqrt{x}}$

For this example we are again faced with a radical in the denominator. However, if we select to proceed as we did in the previous example, we will not rid the denominator of its radical expression.

$$\frac{3}{4+\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{3\sqrt{x}}{\sqrt{x}(4+\sqrt{x})} = \frac{3\sqrt{x}}{(4\sqrt{x}+\sqrt{x^2})} = \frac{3\sqrt{x}}{(4\sqrt{x}+x)}$$

The presence of addition or subtraction in the denominator's quantity indicates that we must select another appearance for the value of one. This expression is called the "conjugate".

Conjugate: Let $a,\ b \in \mathbb{R}$ and x a variable quantity. Then, for the expression, $a+bx$, the expression, $a-bx$, is called the conjugate of $a+bx$.

(The significance of the conjugate becomes important in the next unit as well when we examine its use on Complex Number expressions. For this unit we examine its significance for real numbers.)

For this example, we will use our knowledge of the conjugate and the method of FOIL to simplify the expression.

$$\frac{3}{4+\sqrt{x}}$$

The conjugate of $4+\sqrt{x}$ is $4-\sqrt{x}$, therefore,

$$\frac{3}{(4+\sqrt{x})} \cdot \frac{(4-\sqrt{x})}{(4-\sqrt{x})} = \frac{3(4-\sqrt{x})}{(4+\sqrt{x})(4-\sqrt{x})} = \frac{12-3\sqrt{x}}{16-x}$$

Since there is no common factor for the numerator and denominator, and there is no radical variable in the denominator, the above expression is simplified.

The process for simplifying radical expressions, reviewed above, should be familiar from your studies in algebra. The assignment for this unit will include more practice with this process.

Restricting Values of Radical Quantities

Recall from a previous unit that $\sqrt{-1}$ is an imaginary and not a real number. Although imaginaries (and complex numbers in general) have real applications, their consideration as solutions to radical expressions is usually not allowed except in specific situations. In the next unit we will consider a few of those situations. For this unit we seek to find only real-valued solutions to our equations. By limiting our solutions to only real number answers, we are imposing an artificial or selected restriction on our problem.

Reconsider the following expression from earlier in the unit:

$$\frac{5}{\sqrt{x}}$$

From our last unit, we know that $x \neq 0$ otherwise we obtain division by zero. However, for any $x < 0$, we obtain imaginary numbers for our expression such as:

Let $x = -4$, then $\frac{5}{\sqrt{x}} = \frac{5}{\sqrt{-4}} = \frac{5}{\sqrt{-1 \cdot 4}} = \frac{5}{\pm 2i}$

Because of this, we restrict $x \geq 0$ in order to obtain real answers.

Example #1: Restrict the radicand of the following expression to obtain real number solutions.

$$\sqrt{x-4}$$

For this expression, selecting only positive values for x is not sufficient. If we select $x = 2 > 0$ we obtain:

$$\sqrt{x-4} = \sqrt{2-4} = \sqrt{-2} = \pm i\sqrt{2} \notin \mathbb{R}$$

Instead we proceed, by not simply restricting the variable x, but by restricting the entire radicand in the following manner.

$$x - 4 \geq 0 \quad \Rightarrow \quad x \geq 4$$

Because the expression is not in the denominator of a fraction, we can allow for the radicand to be equal to, or greater than zero, both of which provide real answers.

Example #2: Restrict the radicand of each of the following expressions to obtain real number solutions.

A.) $\sqrt{2x-3}$

$2x-3 \ge 0$

$x \ge \frac{3}{2}$

B.) $\frac{3}{\sqrt{4-x}}$

$4-x > 0$

$x < 4 \quad (\text{Note: } x \ne 4)$

C.) $\frac{5+x}{6-\sqrt{x+1}}$

$x+1 \ge 0$

$x \ge -1$

and

$6-\sqrt{x+1} \ne 0$

$6 \ne \sqrt{x+1}$

$36 \ne x+1$

$x \ne 35$

For "**Part C**" above, we notice the following:

1. If $x+1=0$, the expression is still defined because this does not result in division by "0".

2. We are not concerned about values that cause the numerator to equal zero since "0" can be divided by any number except "0".

3. If $x = 35$, the denominator becomes: $6-\sqrt{35+1} = 6-\sqrt{36} = 6-6 = 0$. Because of the restriction that $x \ge -1$, we do not consider $\sqrt{36} = -6$.

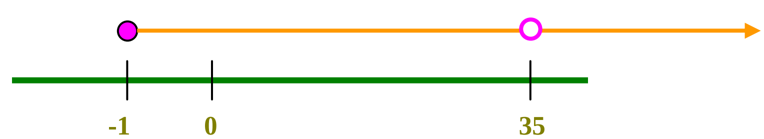
"Part C" is a multiple restriction problem. In a previous unit, we saw multiple restrictions on single values such as:

$$\frac{1}{x}+\frac{1}{x+1} \quad \Rightarrow \quad x \ne 0, \ -1$$

In **"Part C",** however, we have single and multiple restrictions on the problem.

$$x \ge -1 \quad \& \quad x \ne 35$$

These restrictions can be seen on a number line and is a common method for identifying solution sets to radical expressions.



For all values under the arrow from –1 and excluding 35 the expression

$\frac{5+x}{6-\sqrt{x+1}}$ returns real solutions.

Testing for Intervals of Solution

Example #1: Restrict the expression: $\sqrt{x+6}-\sqrt{2-x}$ for real number solutions.

Step #1: Restrict each term individually to real number solutions.

$$\sqrt{x+6} \quad \Rightarrow \quad x+6 \geq 0 \quad \Rightarrow \quad x \geq -6$$
$$\sqrt{2-x} \quad \Rightarrow \quad 2-x \geq 0 \quad \Rightarrow \quad x \leq 2$$

Step #2: Graph the values obtained on a number line.

Step #3: Select numbers to the left, the right and between the numbers and test them in the original expression.

If the number tested results in either or both terms to yield non-real solutions, then the entire interval can be discarded.

Be sure to test the values obtained (–6 and 2) as well.

Test #1: Let $x=-7$: (the interval to the left of $x=-6$)

$$\sqrt{-7+6}-\sqrt{2-(-7)}=\sqrt{-1}-\sqrt{9}$$
$$\Rightarrow \qquad \sqrt{-1} \notin \mathbb{R}$$

Therefore $x \nless -6$

Test #2: Let $x=-6$:

$$\Rightarrow \quad \sqrt{-6+6}-\sqrt{2-(-6)}=\sqrt{0}-\sqrt{8}=-2\sqrt{2} \in \mathbb{R}$$

Therefore $x=-6$ is a possible solution.

Test #3: Let $x = 0$: (the interval between -6 and 2)

$\Rightarrow \quad \sqrt{0+6} - \sqrt{2-0} = \sqrt{6} - \sqrt{2} \in \mathbb{R}$

Therefore $x \geq -6$ are possible solutions.

Test #4: Let $x = 2$:

$\Rightarrow \quad \sqrt{2+6} - \sqrt{2-2} = \sqrt{8} - 0 \in \mathbb{R}$

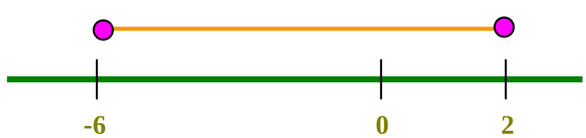
Therefore $x = 2$ is a possible solution.

Test #5: Let $x = 3$: (the interval to the right of 2)

$\Rightarrow \quad \sqrt{3+6} - \sqrt{2-3} = \sqrt{9} - \sqrt{-1} \notin \mathbb{R}$

Therefore $x \not> 2$

Step #4: Graph solutions and state results in set or interval notation.



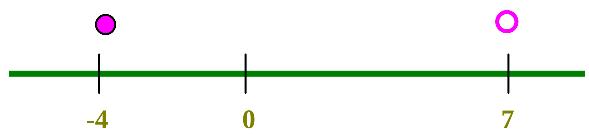
$$A = \left\{ x \middle| -6 \leq x \leq 2 \right\} \quad \text{or} \quad x \in [-6,\ 2]$$

Example #2: Restrict the expression: $\dfrac{\sqrt{x+4}}{\sqrt{x-7}}$ for real number solutions.

Step #1: Restrict each term individually to real number solutions.

$$\begin{aligned} x+4 \geq 0 \quad &\Rightarrow \quad x \geq -4 \\ x-7 > 0 \quad &\Rightarrow \quad x > 7 \\ x \neq 7 & \end{aligned}$$

Step #2: Graph the values obtained on a number line.



Step #3: Select numbers to the left, the right, and between the numbers, and test them in the original expression.

Test #1: Let $x = -5$

$$\Rightarrow \quad \frac{\sqrt{-5+4}}{\sqrt{-5-7}} = \frac{\sqrt{-1}}{\sqrt{-12}} \notin \mathbb{R}$$

Test #2: Let $x = -4$

$$\Rightarrow \quad \frac{\sqrt{-4+4}}{\sqrt{-4-7}} = \frac{\sqrt{0}}{\sqrt{-11}} \notin \mathbb{R}$$

Test #3: Let $x = 0$

$$\Rightarrow \quad \frac{\sqrt{0+4}}{\sqrt{0-7}} = \frac{\sqrt{4}}{\sqrt{-7}} \notin \mathbb{R}$$

Test #4: Let $x = 8$

$$\Rightarrow \quad \frac{\sqrt{8+4}}{\sqrt{8-7}} = \frac{\sqrt{12}}{\sqrt{1}} \in \mathbb{R}$$

Step #4: Graph results. $x \in (7,\ \infty)$

Other Multiple Restrictions to Radical Expressions

Consider the following expression:

$$\sqrt{x-4}+\sqrt{x+2}$$

For each term in the expression, we can find the following restrictions for real number solutions.

$$\begin{aligned} x-4 \geq 0 \quad &\Rightarrow \quad x \geq 4 \\ x+2 \geq 0 \quad &\Rightarrow \quad x \geq -2 \end{aligned}$$

For the second expression if we select $x=0$, then the expression becomes:

$$\sqrt{0+2}=\sqrt{2} \in \mathbb{R}$$

But for the first expression the result is:

$$\sqrt{0-4}=\sqrt{-4}=\pm 2i \notin \mathbb{R}$$

Because both radical terms are combined by addition to form one radical expression, restrictions on the entire expression as a whole must be considered rather than for each term individually. Determining the overall restriction to the expression involves the process of “testing for intervals of solution”.

Testing Quadratic Radicals

Example #1: Restrict the expression $\sqrt{x^2 - 4}$ for real-valued solutions.

For a problem of this type we can proceed as follows:

Step #1: $x^2 - 4 \geq 0$, however because this is a difference of two squares, the expression is factored.

$$(x+2)(x-2) \geq 0$$

We must now consider that the product of two quantities must be greater than or equal to zero. Recall from arithmetic the laws of signed numbers under multiplication.

If $A \cdot B \geq 0$, then $A \geq 0$ and $B \geq 0$ or $A \leq 0$ and $B \leq 0$.

This implies that for $A = x + 2$ and $B = x - 2$, we have:

Case #1

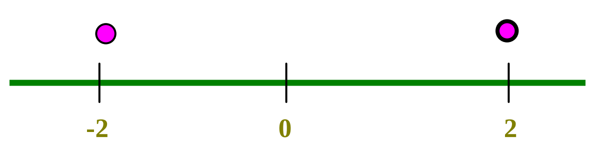
$$\begin{array}{ccc} x+2 \geq 0 & & x-2 \geq 0 \\ & \textit{and} & \\ x \geq -2 & & x \geq 2 \end{array}$$

Case#2

$$\begin{array}{ccc} x+2 \leq 0 & & x-2 \leq 0 \\ & \textit{and} & \\ x \leq -2 & & x \leq 2 \end{array}$$

In either situation we can still proceed with step #2.

Step #2: Graph the values obtained on a number line.



Step #3: Select numbers to the left, the right and between the numbers and test them in the original expression.

Test #1: Let: $x = -3$

$$\Rightarrow \quad (-3+2)(-3-2) = (-1)(-5) = 5 \geq 0 \in \mathbb{R}$$

Test #2: Let: $x = -2$

$$\Rightarrow \quad (-2+2)(-2-2) = (0)(-4) = 0 \geq 0 \in \mathbb{R}$$

Test #3: Let: $x = 0$

$$\Rightarrow \quad (0+2)(0-2) = (2)(-2) = -4 \not\geq 0 \not\in \mathbb{R}$$

Test #4: Let: $x = 2$

$$\Rightarrow \quad (2+2)(2-2) = (4)(0) = 0 \geq 0 \in \mathbb{R}$$

Test #5: Let: $x = 3$

$$\Rightarrow \quad (3+2)(3-2) = (5)(1) = 5 \geq 0 \in \mathbb{R}$$

Step #4: Graph results. $x \in (-\infty, -2] \cup [2, \infty)$