SOLVING RATIONAL EXPRESSIONS AND PARTIAL FRACTION DECOMPOSITION

In previous algebra courses, you learned various techniques for solving a variety of types of equations. The solutions to an equation often take on many values. When the equations involve rational quantities (fractional expressions with variable denominators), there may not only be one or many solutions, but infinite sets of “non-solutions” where certain values cause the equation to take on illegal values. In this unit, we will review techniques for solving rational equations, and then examine a process (which later becomes useful in Calculus) for separating a rational expression into its constituent components. We will also define the meaning of “Degree of a Polynomial” and how it relates to the “Fundamental Theorem of Algebra”.

Polynomial Equations

Polynomial: Let, $A=\{a_0,\ a_1,\ a_2,\ a_3,\ ....,\ a_n,\ ...\}\in\mathbb{R}$ be a set of Real numbers and $X=\{x_0,\ x_1,\ x_2,\ x_3,\ ...,\ x_n,\ ...\}\in\mathbb{R}$ be a set of real variables. Let, $\{0\}\cup\{n_1,\ n_2,\ n_3,\ ...n_i,\ ...\}\in\mathbb{N}$ be the set of Natural numbers unioned with the integer $\{0\}$. Then the expression:

$$a_0x_0^n + a_1x_1^{n-1} + a_2x_2^{n-2} + \ ...\ + a_{n-2}x_{n-2}^2 + a_{n-1}x_{n-1}^1 + a_nx_n^0$$

is a Polynomial of Degree $=n$

In functional notation a polynomial is denoted: $P_0(x)$

The significant aspects of this definition are:

1.) All values in the expression are Natural or Real.
2.) All variable exponents are natural or zero.
   - This indicates that no variable quantity can be found in the denominator of any fractional expression.
   - i.e. $x^{-3}=\dfrac{1}{x^3}$, therefore, x^{-3} is not a polynomial.

Degree of a Polynomial:

Let $P_0(x)=a_0x_0^n + a_1x_1^{n-1} + a_2x_2^{n-2} + \ ...\ + a_{n-1}x_{n-1}^1 + a_nx_n^0$ be a polynomial. Then, for $n>n-1>n-2>\ ...\ >2>1>0$, the degree of $P_0(x)=n$.

Denoted: $dP_0(x)$

Example #1: Let $P_0(x)=\dfrac{3}{4}x^5-6x^7+\dfrac{21}{11}x^8$.

Since $8>7>5$, then $dP_0(x)=8$.

For polynomials of more than one variable quantity, the degree of the polynomial is the highest sum total of exponents on any one term.

Example #2: Let $W_o(xy)=4x^2y^3-6x^2y-13x^3y^3$, find $dW_0(xy)$

For each term in the polynomial:

A.) The degree is:

1.) $4x^2y^3$: sum of exponents $= 2 + 3 = 5$
2.) $-6x^2y$: sum of exponents $= 2 + 1 = 3$
3.) $-13x^3y^3$: sum of exponents $= 3 + 3 = 6$

-Therefore $dW_0(xy) = 6$

B.) The degree for each variable can also be established.

1.) For $-13x^3y^3$, the exponent on $x = 3$
2.) For $-13x^3y^3$, & $4x^2y^3$, the exponent on $y = 3$

-Therefore $d_xW_0(xy) = 3$ & $d_yW_0(xy) = 3$

The degree of a polynomial is an important tool in determining possible solutions to an equation. The concept's significance was highlighted in the year 1799 when the German mathematician, Karl Friedrich Gauss (1777-1855), used it as a result of his proof of the "Fundamental Theorem of Algebra".

Theorem: The Fundamental Theorem of Algebra (FTA)

Let $P_0(x)$ be a polynomial of degree, n where $n \in \mathbb{N}$. Then the equation, $P_0(x) = 0$ has exactly 'n' solutions where each solution $x_i \in \mathbb{R}$ or $x_i \in \mathbb{C}$.

*Note: For the purposes of this unit, we will not consider solutions from $\mathbb{C}$, the complex numbers. Complex numbers will be discussed in a later unit.

The significance of the FTA is that, given an equation of known degree 'n', there will be at most 'n' real solutions to the equation.

Example #3: Determine the number of real solutions to the following:

a.) $P_0(x) = 3x + 4$

$dP_0(x) = 1$

Verify: Let, $P_0(x) = 0 \quad \Rightarrow \quad 3x + 4 = 0 \quad \Rightarrow \quad x = -\frac{4}{3}$ (**One** solution)

b.) $Q_0(x) = x^2 - 4$

$$dQ_0(x) = 2$$

Verify: Let,

$$\begin{aligned} Q_0(x) = 0 \quad &\Rightarrow \quad x^2 - 4 = 0 \\ &\Rightarrow \quad x^2 = 4 \\ &\Rightarrow \quad x = \pm 2 \end{aligned} \quad \text{(\textbf{Two} solutions)}$$

c.) $W_0(x) = 9x^2 - 12x + 4$

$$dW_0(x) = 2$$

Verify: Let,

$$\begin{aligned} W_0(x) = 0 \quad \Rightarrow \quad 9x^2 - 12x + 4 &= 0 \\ (3x-2)(3x-2) &= 0 \\ (3x-2)^2 &= 0 \\ x &= \frac{2}{3} \end{aligned} \quad \text{(\textbf{One} Real solution)}$$

Rational Expressions and Equations

By our definition of Polynomial, we know that all exponents on any polynomial's variable quantities must be zero or natural numbers. However, not all problems involve equations of this type. For problems that involve denominators of variable quantity, we need to expand our solution techniques to account for **non-solutions** to the problems as well. Equations of this type are called "Rational Equations".

Rational "Polynomial" Expression

Let $P_0(x),\ \ Q_0\left(x\right) \in \mathbb{R}$, be real valued polynomials of degree n & m, respectively, $(n \le m \ \text{ or } \ n > m)$. Then the expression, $\dfrac{P_0(x)}{Q_0(x)}:\ \ Q_0(x) \neq 0$, is a 'Rational Expression' of the polynomials $P_0(x),\ \ \&\ \ Q_0(x)$.

As we know, division by zero is undefined. Therefore, in a rational expression, those values that cause the denominator to equal zero must be identified and discarded as possible values in any variable quantity. As first mentioned in our discussions of functions, we will "restrict" those values from our expressions or equations.

Example #1: Restrict illegal values for any variable in each of the following.

a.) Given, $\dfrac{3x+5}{7x-8}$, restrict all values of x that would result in division by zero.

For this problem we want to identify and solve where $7x-8=0$, then disallow our result from the expression.

$$\Rightarrow \quad x \neq \frac{8}{7}$$

b.) Given, $3x - \dfrac{2x^2+5x-9}{x^2+7x+6}$, restrict all values of x that would result in division by zero.

For this problem we want to identify and solve where $x^2+7x+6=0$, and then disallow our results from the expression.

$$\Rightarrow \quad x^2 + 7x + 6 \neq 0$$

$$(x+6)(x+1) \neq 0$$

$$x \neq -1, \ -6$$

c.) Given, $\dfrac{37x^3 - 4x + 11}{x^2 + 3}$, restrict all values of x that would result in division by zero.

For this problem we want to identify and solve where $x^2 + 3 = 0$, then disallow our results from the expression.

$$\Rightarrow \quad x^2 + 3 \neq 0$$

$$x^2 \neq -3$$

$$x \neq \pm\sqrt{-3}$$

$$x \neq \pm i\sqrt{3} \in \mathbb{C}$$

Since the only restrictions for this expression occur in the domain of the complex numbers, we say there are no restrictions on x for $x \in \mathbb{R}$.

Solving Rational Equations (Review)

The following procedure will serve as a review for solving rational equations before we proceed to the process of Partial Fraction Decomposition. Although solution techniques for solving rational equations are important to our study, a detailed examination of these processes will not be covered. The assignment for this unit will provide a few practice problems in this area.

Procedure for Solving Rational Expressions

1. Factor all quantities in the numerators and denominators of all fractions.
2. Restrict all values of *x* from every denominator that would result in division by zero.
3. For multiplication and division equations:
   a. Cancel any identical quantities found in the numerator and denominator.
   b. Cross-multiply and solve.
4. For addition and subtraction equations:
   a. Identify the common denominator of all terms for both sides of the equals sign.
   b. Multiply every term on both sides by the common denominator as a fraction with denominator equal to 'one'.
   c. Cancel all quantities found in common with the common denominator and with the denominators of each fraction, and then solve the remaining equation.

*Note: Step #2 in this procedure must be performed before step #3 or step #4. Canceling variable quantities found in the denominators of a rational equation prior to restricting may result in solutions that are illegal.

Example #1: Solve: $\dfrac{x^2+x-2}{x+2}=-3$

Step #1: Factor all quantities.

$$\frac{(x+2)(x-1)}{(x+2)}=-3$$

Step #2: Restrict values of x in the denominator.

$$x+2\neq 0 \quad \Rightarrow \quad x\neq -2$$

Step #3: Cancel and solve.

$$\frac{(x\cancel{+}2)(x-1)}{(x\cancel{+}2)}=-3 \quad\Rightarrow\quad x-1=-3 \quad\Rightarrow\quad x=-2$$

Therefore, no solution, or $\varnothing$ since: $x \neq -2$.

*Note: Had we performed step #3 before step #2, we would have obtained an answer, $x=-2$, which is prohibited from the problem.

- All restrictions apply to the original equation prior to cancellation and not to any intermediate step.

Example #2: Solve: $\dfrac{x}{x-3}+\dfrac{2x}{x+3}=\dfrac{18}{x^2-9}$

Step #1: Factor all quantities.

$$\frac{x}{x-3}+\frac{2x}{x+3}=\frac{18}{(x+3)(x-3)}$$

Step #2: Restrict values of x in the denominator.

$$x-3\neq 0 \quad\Rightarrow\quad x\neq 3$$
$$x+3\neq 0 \quad\Rightarrow\quad x\neq -3$$

Step #3: Multiply every term on both sides by the common denominator (c.d.) as a fraction with denominator equal to 'one'.

$$\text{C.D.}=(x+3)(x-3)$$

$$\frac{x}{(x\cancel{-}3)}\cdot\frac{(x+3)(x\cancel{-}3)}{1}+\frac{2x}{(x\cancel{+}3)}\cdot\frac{(x\cancel{+}3)(x-3)}{1}=\frac{18}{(x\cancel{+}3)(x\cancel{-}3)}\cdot\frac{(x\cancel{+}3)(x\cancel{-}3)}{1}$$

$$\begin{aligned}\Rightarrow\quad x(x+3)+2x(x-3)&=18\\ x^2+3x+2x^2-6x&=18\\ 3x^2-3x-18&=0\\ x^2-x-6&=0\\ (x+2)(x-3)&=0 \quad\Rightarrow\quad x=-2,\ \cancel{3}\end{aligned}$$

Partial Fraction Decomposition

When we examine the expression:

$$\frac{8x-1}{x^2+x-6}$$

We note that the degree of the numerator equals '1' and the degree of the denominator equals '2'.

If we examine the expression:

$$\frac{5}{x+3}+\frac{3}{x-2}$$

We note that the degree of the numerator equals '0' and the degree of each denominator equals '1'.

(Note: $5=5x^0 \quad : \quad 3=3x^0$).

Both examples are equivalent forms of the same expression:

$$\frac{5}{x+3}+\frac{3}{x-2}=\frac{8x-1}{x^2+x-6}$$ (Verify this on your own.)

As a rule, expressions of lesser degree are less complicated and generally less tedious to manipulate. Therefore, it is useful to find a method that separates or "decomposes" an expression of higher degree into equivalent expressions of lower degree. In Calculus this process will allow you to find an "Integral" of a function, piece by piece, more easily than if you were to attempt to deal with the entire expression. The process for finding the constituent quantities of a rational expression is called **"Partial Fraction Decomposition"** and its use is an important tool in the study if higher mathematical constructs.

Theorem: Partial Fraction Decomposition

Let, $P_0(x),\ Q_0(x)\in\mathbb{R}$ be real polynomials where $dP_0(x)=n,\ dQ_o(x)=m \quad \& \quad n<m$.

If $(ax+b)$ is a factor of $Q_0(x)$, then for, $R_0(x)=\dfrac{P_0(x)}{Q_0(x)}$, (is a rational polynomial expression), the Partial Fraction Decomposition of $R_0(x)$ contains a term of the form, $\dfrac{A}{ax+b}$.

This sophisticated theorem simply states that, any rational expression can be represented as a sum of two or more fractions. For this method, we seek to find those "partial fractions" that composed the original sum of rational expressions.

*Note: The degree of the numerator's polynomial must be less than the degree of the denominator in order for this process to proceed, from above, $n < m$.

Procedure for Partial Fraction Decomposition

1. Factor all quantities in the expression.
2. List a sum of fractions whose denominators contain one of each of the factors found in step #1 and whose numerators contain the unknown quantities of A, B, C, …
3. Multiply every term by the C.D. of all terms to clear the expression of all fractions.
4. Solve for A, B, C, … by substitution of values to make the variable multipliers of A, B, C, … equal to zero.
5. Verify

Example #1: Find the partial fraction decomposition for:

$$\frac{x}{x^2-x-6}$$

Step #1: Factor all quantities.

$$\frac{x}{x^2-x-6}=\frac{x}{(x-3)(x+2)}$$

Step #2: List a sum of fractions for each factored quantity.

$$\frac{x}{(x-3)(x+2)}=\frac{A}{(x-3)}+\frac{B}{(x+2)}$$

Step #3: Multiply all terms by the C.D. to clear the expression of fractions.

$$\text{C.D.}=(x-3)(x+2)$$

$$\frac{\cancel{(x-3)}\cancel{(x+2)}}{1}\cdot\frac{x}{\cancel{(x-3)}\cancel{(x+2)}}=\frac{\cancel{(x-3)}(x+2)}{1}\cdot\frac{A}{\cancel{(x-3)}}+\frac{(x-3)\cancel{(x+2)}}{1}\cdot\frac{B}{\cancel{(x+2)}}$$

$$x=A(x+2)+B(x-3)$$

Step #4: Substitute values for x to make the variable multipliers of A & $B=0$, individually. Then solve for the remaining variable.

a.) Let $x=-2 \quad \Rightarrow$

$$
\begin{aligned}
-2 &= A(-2+2)+B(-2-3) \\
-2 &= A(0)-5B \\
-2 &= -5B \\
\frac{2}{5} &= B
\end{aligned}
$$

b.) Let $x=3 \quad \Rightarrow$

$$
\begin{aligned}
3 &= A(3+2)+B(3-3) \\
3 &= 5A-0\cdot B \\
3 &= 5A \\
\frac{3}{5} &= A
\end{aligned}
$$

Step #5: Verify:

$$\frac{x}{(x-3)(x+2)}=\frac{3/5}{(x-3)}+\frac{2/5}{(x+2)}$$

Multiply by C.D.

$$
\begin{aligned}
x &= \frac{3}{5}(x+2)+\frac{2}{5}(x-3) \\
5x &= 3(x+2)+2(x-3) \\
5x &= 3x+6+2x-6 \\
5x &= 5x \\
x &= x \quad \text{(Verified)}
\end{aligned}
$$

Example #2: Find the partial fraction decomposition for:

$$\frac{8x-22}{6x^2+17x-14}$$

Step #1: Factor all quantities.

$$\frac{8x-22}{6x^2+17x-14}=\frac{2(4x-11)}{(2x+7)(3x-2)}$$

Step #2: List a sum of fractions for each factored quantity.

$$\frac{2(4x-11)}{(2x+7)(3x-2)}=\frac{A}{(2x+7)}+\frac{B}{(3x-2)}$$

Step #3: Multiply all terms by the C.D. to clear the expression of fractions.

$$\text{C.D.}=(2x+7)(3x-2)$$

$$\frac{\cancel{(2x+7)}\cancel{(3x-2)}}{1}\cdot\frac{2(4x-11)}{\cancel{(2x+7)}\cancel{(3x-2)}}=\frac{\cancel{(2x+7)}(3x-2)}{1}\cdot\frac{A}{\cancel{(2x+7)}}+\frac{(2x+7)\cancel{(3x-2)}}{1}\cdot\frac{B}{\cancel{(3x-2)}}$$

$$2(4x-11)=A(3x-2)+B(2x+7)$$

Step #4: Substitute values for x to make the variable multipliers of A & $B=0$, individually. Then solve for the remaining variable.

a.) Let $x=\dfrac{2}{3}$ $\Rightarrow$ $2(4x-11)=A(3x-2)+B(2x+7)$

$$2\left(4\left(\frac{2}{3}\right)-11\right)=A\left(3\left(\frac{2}{3}\right)-2\right)+B\left(2\left(\frac{2}{3}\right)+7\right)$$

$$\frac{16}{3}-22=0\cdot A+\left(\frac{4}{3}+7\right)B$$

$$\frac{-50}{3}=\frac{25}{3}B$$

$$-50=25B$$

$$-2=B$$

b.) Let $x = \frac{-7}{2} \quad \Rightarrow \quad 2(4x-11) = A(3x-2) + B(2x+7)$

$$2\left(4\left(\frac{-7}{2}\right)-11\right) = A\left(3\left(\frac{-7}{2}\right)-2\right) + B\left(2\left(\frac{-7}{2}\right)+7\right)$$

$$-28-22 = \frac{-25}{2}A + 0\cdot B$$

$$-50 = \frac{-25}{2}A$$

$$-100 = -25A$$

$$4 = A$$

Step #5: Verify:

$$\frac{2(4x-11)}{(2x+7)(3x-2)} = \frac{4}{(2x+7)} + \frac{-2}{(3x-2)}$$

Multiply by C.D.

$$8x-22 = 4(3x-2) - 2(2x+7)$$
$$8x-22 = 12x-8-4x-14$$
$$8x-22 = 8x-22$$

(Verified)