MATRICES AND SYSTEMS OF EQUATIONS

Matrices are useful to mathematics and in real world applications as well. Computer spreadsheet programs essentially operate as interactive matrices. Matrices are used to describe multiple electronic circuits. When the elements of a matrix are complex numbers, then they can be used to evaluate and describe a wide range of topics from astronomy to banking decisions, and even used to develop automotive paint mixtures that are highly resistant to scratches and other types of abuse. In this unit, we will explore inverse matrices and use them to solve systems of linear equations. In a real world situation, a set of solutions to a system of related events is where matrix applications begin.

The Identity and Inverse Matrices

The Determinant of a Matrix

Using the Calculator to find the Determinant of a Square Matrix

Solving Systems with Matrix Equations

The Identity and Inverse Matrices

A matrix can be used to encode a message and another matrix, **it's inverse**, is used to decode a message once it is received.

Recall that a **square matrix** is a matrix that has the same number of rows and columns.

Example #1:

$$[A]_{2\times 2} = \begin{bmatrix} 2 & 3 \\ 1/2 & 3/4 \end{bmatrix}_{2\times 2} \qquad [B]_{3\times 3} = \begin{bmatrix} 4 & -9 & 5 \\ 7 & -3 & 8 \\ 0 & 2 & -1 \end{bmatrix}_{3\times 3} \qquad [C]_{4\times 4} = \begin{bmatrix} -2 & -8 & 4 & 8 \\ 0 & 1 & -3 & 6 \\ 4 & -4 & 0 & 9 \\ 11 & 32 & 57 & 3 \end{bmatrix}_{4\times 4}$$

Dimensions: 2×2 $\qquad$ 3×3 $\qquad$ 4×4

Identity Matrix for Multiplication

Let $[A]$ be a square matrix with n rows and n columns. Let $[I]$ be a matrix with the same dimensions and with 1's on the main diagonal and 0's elsewhere. (The "main diagonal" of a matrix starts in the upper left and ends in the lower right of a square matrix).

Then, $[A]\cdot[I] = [I]\cdot[A] = [A]$.

$$[I]_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [I]_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [I]_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices perform the same function for matrix multiplication that the number '1' performs for multiplication of single numbers. The product of any real number and '1' is the same number. The product of a square matrix $[B]$ and its identity $[I]$ is the matrix $[B]$.

Example #2:

$$\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-4\cdot 1)+(4\cdot 0) & (-4\cdot 0)+(4\cdot 1) \\ (2\cdot 1)+(8\cdot 0) & (2\cdot 0)+(8\cdot 1) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}$$

$$[\mathrm{B}]_{2\times 2} \times [\mathrm{I}]_{2\times 2} \qquad = \qquad [\mathrm{B}]_{2\times 2}$$

The Inverse of a Matrix

Let $[\mathrm{A}]_{n\times n}$ be a square matrix with n rows and n columns. If there is an $n \times n$ matrix $[\mathrm{B}]_{n\times n}$ such that $[\mathrm{A}]_{n\times n} \times [\mathrm{B}]_{n\times n} = [\mathrm{B}]_{n\times n} \times [\mathrm{A}]_{n\times n} = [\mathrm{I}]_{n\times n}$,
then $[\mathrm{A}]_{n\times n}$ and $[\mathrm{B}]_{n\times n}$ are inverse matrices. The inverse of matrix $[\mathrm{A}]_{n\times n}$ is denoted by $[\mathrm{A}]^{-1}_{n\times n}$.

*(Note: $[\mathrm{A}]^{-1} \neq \dfrac{1}{[\mathrm{A}]}$). The product of a real number and its multiplicative inverse is 1. The product of a square matrix and its inverse is the identity matrix $[\mathrm{I}]_{n\times n}$).

Example #3: Let: $[\mathrm{C}] = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ and $[\mathrm{D}] = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$

To show that $[\mathrm{C}]$ and $[\mathrm{D}]$ are inverses of one another, multiply $[\mathrm{C}]\,[\mathrm{D}]$ and $[\mathrm{D}]\,[\mathrm{C}]$.

$$[\mathrm{C}]\,[\mathrm{D}] = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (2\cdot -7)+(5\cdot 3) & (2\cdot 5)+(5\cdot -2) \\ (3\cdot -7)+(7\cdot 3) & (3\cdot 5)+(7\cdot -2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\mathrm{D}]\,[\mathrm{C}] = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} (-7\cdot 2)+(5\cdot 3) & (-7\cdot 5)+(5\cdot 7) \\ (3\cdot 2)+(-2\cdot 3) & (3\cdot 5)+(-2\cdot 7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Determinant of a Matrix

Related to our study of the inverse and identity matrix are ways to determine if a matrix problem has a solution and finding methods for finding those solutions. Our first investigation into matrix problems involves the determinant of the matrix. Each square matrix can be assigned a real number which is called the determinant.

Determinant of a 2 x 2 Matrix

Let $[\mathrm{A}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the **determinant** of $[\mathrm{A}]$, denoted by $\det[\mathrm{A}]$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined as

$$\det[\mathrm{A}] = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Theorem #1: Matrix A has an inverse, if and only if, $\det[\mathrm{A}] \neq 0$. Therefore, unlike real numbers, not all matrices have an inverse.

Using the Determinant to find the Inverse of a Square Matrix

Procedure:

1) Find the difference of the cross products.
2) Find the reciprocal of this number and multiply it times the matrix as a scalar.
3) Change the location of *a* and *d* in the matrix.
4) Change the signs of *b* and *c* in the matrix.

$$[\mathrm{A}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad [\mathrm{A}]^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example #1: Find the determinant, and then find the inverse of the following matrix:

$$[\mathrm{B}] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Step #1: Find the Determinant.

$$\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 1\times 5 - 2\times 3 = 5 - 6 = -1$$

*Step #2***:** Multiply the determinant times the adjusted matrix according to the above procedure.

$$[\mathrm{B}]^{-1} = -1\cdot\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

*To find the inverse of a 3 × 3 matrix, we will use our calculator; however, the description for finding higher order determinants is worth noting as shown below.

Determinant of a 3 x 3 Matrix

Let $[\mathrm{A}] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Then the **determinant** of $[\mathrm{A}]$, denoted by $\det[\mathrm{A}]$ or $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, is defined as:

$$\det[\mathrm{A}] = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

This definition becomes especially significant in the study of Complex Numbers in higher math such as Calculus.

Using the Calculator to find the Determinant of a Square Matrix

Example #1: Find the determinant of the following matrix:

$$[\mathrm{A}]_{3\times 3} = \begin{bmatrix} 12 & 7 & 1 \\ 3 & 4 & 2 \\ 5 & 3 & -3 \end{bmatrix}$$

Procedure: Find the determinant:

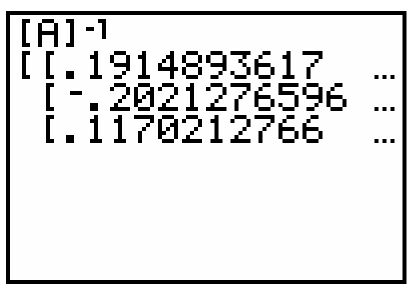
1.) Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$ to view the matrix features. Enter data into matrix $[\mathrm{A}]$.
(Recall: Edit / [A] , then $\boxed{\text{2nd}}$ QUIT)

2.) Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$. Cursor right to the 'MATH' menu and press $\boxed{\text{ENTER}}$.

3.) Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, and then $\boxed{\text{ENTER}}$. " det($[\mathrm{A}]$) " should now be on your screen.
Press $\boxed{\text{ENTER}}$. The answer should be -94.

Find the inverse of a Square Matrix on your calculator:

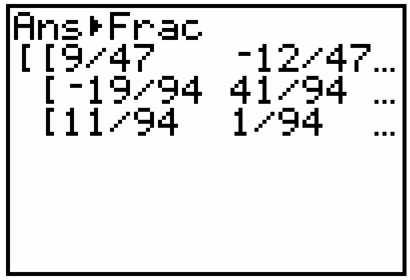
1.) Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, $\boxed{\text{ENTER}}$,then, $\boxed{x^{-1}}$ again. Your screen should display $[\mathrm{A}]^{-1}$.
Press $\boxed{\text{ENTER}}$. Your calculator should display the following:



Which if you scroll with your right arrow records all the following values:

$$[\mathrm{A}]^{-1} = \begin{bmatrix} .19148... & -.25531... & -.10638... \\ -.20212... & .43617... & .22340... \\ .11702... & .01063... & -.28723... \end{bmatrix}$$

2.) To view the inverse as fractions press $\boxed{\text{MATH}}$, $\boxed{\text{ENTER}}$, $\boxed{\text{ENTER}}$. Your calculator should display the following:



Which if you scroll with your right arrow records all the following values:

$$[A]^{-1} = \begin{bmatrix} \frac{9}{47} & \frac{-12}{47} & \frac{-5}{47} \\ \frac{-19}{94} & \frac{41}{94} & \frac{21}{94} \\ \frac{11}{94} & \frac{1}{94} & \frac{-27}{94} \end{bmatrix}$$

Solving Systems with Matrix Equations

A system of linear equations can be written as a matrix equation.

Example #1: Write the following system as a coefficient matrix, a variable matrix, and a constant matrix.

$$\begin{aligned} 2a+4b&=-3 \\ a-b&=9 \end{aligned}$$

Coefficient Matrix A	Variable Matrix X	Constant Matrix B
$[\mathrm{A}]=\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$	$[\mathrm{X}]=\begin{bmatrix} a \\ b \end{bmatrix}$	$[\mathrm{B}]=\begin{bmatrix} -3 \\ 9 \end{bmatrix}$

From our previous discussion, we examined how the inverse of a matrix plays the same role in matrix multiplication that multiplicative inverses play in solving algebraic equations. For example, when solving the equation, $3x=7$ for x, we usually proceed by dividing through by 3, and often overlook the fact that the actual process involves multiplying both sides of the equation by the multiplicative inverse of 3. The actual process is:

$$\begin{aligned} 3x&=7 \\ \frac{1}{3}\cdot 3x&=\frac{1}{3}\cdot 7 \\ x&=\frac{7}{3} \end{aligned}$$

When solving a system of equations in a matrix, we precede in much the same manner. Both sides of the equation will be multiplied by the inverse of the **coefficient matrix.** This will achieve the same result that multiplying the above equation by the inverse of 3 had on that equation. In the above equation's solution, the coefficient on the variable x became'1'. When we multiply a **coefficient matrix** by its inverse, all the coefficients on the variables in the system's solution become '1'. This, then, solves the system.

Procedure for solving a system of equations using matrices:

Example #2: Solve the system, $\begin{aligned} 2a+4b&=-3 \\ a-b&=9 \end{aligned}$, for a & b

Step #1: Find the coefficient matrix, the variable matrix, and the constant matrix.

$$[\mathrm{A}]=\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix} \qquad [\mathrm{X}]=\begin{bmatrix}a\\ b\end{bmatrix} \qquad [\mathrm{B}]=\begin{bmatrix}-3\\ 9\end{bmatrix}$$

Step #2: Find the inverse of the coefficient matrix:

a.) Determinant of $[\mathrm{A}]$:

$$[\mathrm{A}]=\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix} \quad \Rightarrow \quad 2(-1)-4(1)=-6$$

b.) Inverse of $[\mathrm{A}]$:

$$[\mathrm{A}]^{-1}=-\frac{1}{6}\begin{bmatrix}-1 & -4\\ -1 & 2\end{bmatrix}$$

Step #3: Multiply both sides of the equation by $[\mathrm{A}]^{-1}$

$$[\mathrm{A}]\cdot[\mathrm{X}]=[\mathrm{B}]$$

$$[\mathrm{A}]^{-1}\cdot[\mathrm{A}]\cdot[\mathrm{X}]=[\mathrm{A}]^{-1}\cdot[\mathrm{B}]$$

$$-\frac{1}{6}\begin{bmatrix}-1 & -4\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}\cdot\begin{bmatrix}a\\ b\end{bmatrix}=-\frac{1}{6}\begin{bmatrix}-1 & -4\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}-3\\ 9\end{bmatrix}$$

Multiply the matrices on both sides of the equation by the inverse. The result will be the Identity Matrix, $[\mathrm{I}]$, times the variable matrix, $[\mathrm{X}]$ on the left, which simplifies to $[\mathrm{X}]$.

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}a\\ b\end{bmatrix} = -\frac{1}{6}\begin{bmatrix}-33\\ 21\end{bmatrix}$$

Step #4: Simplify the right by scalar multiplication to obtain the final result.

$$\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}11/2\\ -7/2\end{bmatrix}$$

The solution to the system is: $a=\frac{11}{2}, \quad b=\frac{-7}{2}$

Example #3: Solve the 3×3 system:

$$
\begin{aligned}
5x + y - 7z &= 4 \\
x - 3y - 2z &= -2 \\
3x + y - 6z &= 6
\end{aligned}
$$

For this problem we will use the graphing calculator to find the "Inverse of the coefficient matrix".

Step #1: Find the coefficient matrix, the variable matrix, and the constant matrix.

$$[A] = \begin{bmatrix} 5 & 1 & -7 \\ 1 & -3 & -2 \\ 3 & 1 & -6 \end{bmatrix} \qquad [X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad [B] = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$$

Step #2: Find the inverse of the coefficient matrix using the graphing calculator.

a.) Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, then right arrow to the 'EDIT' menu and enter the coefficient matrix into $[A]$.

b.) Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, $\boxed{\text{ENTER}}$, $\boxed{x^{-1}}$, $\boxed{\text{MATH}}$, $\boxed{\text{ENTER}}$, $\boxed{\text{ENTER}}$ (to represent the inverse in fraction form).

$$[A]^{-1} = \begin{bmatrix} 2/3 & -1/30 & -23/30 \\ 0 & -3/10 & 1/10 \\ 1/3 & -1/15 & -8/15 \end{bmatrix}$$

Step #3: Press $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, $\boxed{\text{ENTER}}$, $\boxed{x^{-1}}$, $\boxed{\text{2nd}}$, $\boxed{x^{-1}}$, " 2 ", $\boxed{\text{MATH}}$, $\boxed{\text{ENTER}}$, $\boxed{\text{ENTER}}$ (to represent the solution in fraction form).

Answer:

$$[X] = \begin{bmatrix} -28/15 \\ 6/5 \\ -26/15 \end{bmatrix}$$

*Note: Although the calculator can perform many matrix tasks quickly and without as many steps, the procedures outlined in the last two units to perform matrix calculations are essential to an understanding of higher mathematical operations. The time saving features of the calculator should be regarded as a means to check and verify results.