INTRODUCTION TO MATRICES

As mentioned in the previous unit, finding solutions to a system of equations by elimination of variables is a first step in applying Matrix Algebra to solving systems of equations. In this unit we will discuss the important aspects of how Matrices can be used to represent data and we will discuss the rules for matrix addition, subtraction, and multiplication.

Using Matrices to Represent Data

Special Matrices

Basic Operations on Matrices

Order of Operations

Entering Matrices on the Graphing Calculator

Using Matrices to Represent Data

Matrix: a system of rows and columns that are used as a tool for organizing numbers or data so that each position in the matrix has a purpose.

Each value in a matrix is called an element, such as in the example below:

Example #1: $[A] = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -4 \end{bmatrix}$ ← **Rows**
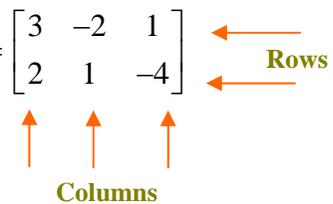
↑ ↑ ↑
Columns

*Matrices are named using their dimensions (rows x columns) therefore the matrix above would be known as a 2 x 3 (read "2 by 3") matrix and is denoted as:

$$[A]_{2\times3}$$

Each element of a matrix has a special location. For example, –2 is in the first row, second column, and would be represented as a_{12}, –4 would be represented as a_{23}.

Special Matrices

Row matrix: only one row → $\begin{bmatrix} 2 & 0 & -7 \end{bmatrix}$

Column matrix: only one column → $\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$

Square matrix: same number of rows and columns → $\begin{bmatrix} 3 & 0 \\ -6 & 4 \end{bmatrix}$ $\quad or \quad$ $\begin{bmatrix} 3 & -9 & 4 \\ 0 & -6 & 3 \\ 5 & 13 & -4 \end{bmatrix}$

Two matrices are considered equal if they have the same dimensions, and each element of one matrix is equal to the corresponding element of the other.

> *Example #1*:
>
> a.) $\begin{bmatrix} x & y \\ w & z \end{bmatrix}_{2\times2} = \begin{bmatrix} 5 & -6 \\ 8 & 0 \end{bmatrix}_{2\times2}$ $\quad \Rightarrow x = 5, \quad y = -6$
> $\Rightarrow w = 8, \quad z = 0$
>
> b.) $\begin{bmatrix} 2x+y \\ x-3y \end{bmatrix}_{2\times1} = \begin{bmatrix} 6 \\ 31 \end{bmatrix}_{2\times1}$
>
> c.) $2\begin{bmatrix} 3-x & 2 \\ -3 & y \end{bmatrix}_{2\times2} = \begin{bmatrix} 5 & 4 \\ -6 & 3y-4 \end{bmatrix}_{2\times2}$
>
> *Since the matrices have the same dimensions and they are equal, the corresponding elements are equal.
>
> When you write the sentences for example "b" above that show this equality, two linear equations are formed. To solve for x and y use either substitution or elimination.

$$2x+y=6 \quad \longrightarrow \quad 3(2x+y=6) \quad \longrightarrow \quad 6x+3y=18$$
$$x-3y=31 \quad\quad\quad\quad x-3y=31 \quad\quad\quad + \quad \underline{x-3y=31}$$
$$7x=49$$
$$x=7$$

Substitute 7 for x in either of the original equations to solve for y.

$$2(7) + y = 6$$
$$14 + y = 6$$
$$y = -8$$

The solution to the system of equations is $(7, -8)$.

For example "c" above, we must first multiply each element of the first matrix by the scalar number 2 before setting up our two equations.

First: $2\begin{bmatrix} 3-x & 2 \\ -3 & y \end{bmatrix}_{2\times 2} \Rightarrow \begin{bmatrix} 2(3-x) & 2(2) \\ 2(-3) & 2(y) \end{bmatrix}_{2\times 2} \Rightarrow \begin{bmatrix} 6-2x & 4 \\ -6 & 2y \end{bmatrix}_{2\times 2}$

Then: $\begin{bmatrix} 6-2x & 4 \\ -6 & 2y \end{bmatrix}_{2\times 2} = \begin{bmatrix} 5 & 4 \\ -6 & 3y-4 \end{bmatrix}_{2\times 2} \Rightarrow \begin{matrix} 6-2x = 5 \\ 2y = 3y-4 \end{matrix} \Rightarrow \begin{matrix} x = 1/2 \\ y = 4 \end{matrix}$

Basic Operations on Matrices

Scalar Multiplication

To multiply a matrix by a constant, each element of the matrix is multiplied individually. The name for this constant is 'scalar'.

Example #1: Multiply matrix $[A]$ by 3. (This is represented as $3 \cdot [A]$)

$$[A]_{3\times2} = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 6 & -4 \end{bmatrix}_{3\times2} \qquad \Rightarrow \qquad 3[A]_{3\times2} = \begin{bmatrix} 6 & 3 \\ -9 & 0 \\ 18 & -12 \end{bmatrix}_{3\times2}$$

*Notice that each element of matrix $[A]$ was multiplied by three.

Adding or Subtracting Matrices

To add or subtract two or more matrices, the matrices must have the same dimensions.

Example #2: Add or subtract the corresponding elements of matrix $[A]$ & $[B]$.

$$[A]_{2\times3} = \begin{bmatrix} 2 & -1 & 8 \\ 4 & 7 & 9 \end{bmatrix}_{2\times3} ; \; [B]_{2\times3} = \begin{bmatrix} -1 & 4 & -3 \\ 7 & 2 & -6 \end{bmatrix}_{2\times3}$$

Answer:
$$\begin{bmatrix} 2 & -1 & 8 \\ 4 & 7 & 9 \end{bmatrix} + \begin{bmatrix} -1 & 4 & -3 \\ 7 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2+-1 & -1+4 & 8+-3 \\ 4+7 & 7+2 & 9+-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 5 \\ 11 & 9 & 3 \end{bmatrix}_{2\times3} = [C]_{2\times3}$$

Order of Operations

If scalar multiplication, and addition or subtraction, occur in the same problem, perform the scalar multiplication first.

$$\textit{Example \#1}:\ 3\begin{bmatrix}4\\1\\7\end{bmatrix}+2\begin{bmatrix}3\\-2\\6\end{bmatrix}-5\begin{bmatrix}-2\\3\\6\end{bmatrix}=\begin{bmatrix}12\\3\\21\end{bmatrix}+\begin{bmatrix}6\\-4\\12\end{bmatrix}+\begin{bmatrix}10\\-15\\-30\end{bmatrix}=\begin{bmatrix}28\\-16\\3\end{bmatrix}$$
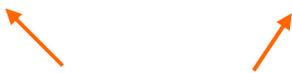
*Notice that the last matrix was multiplied by a (–5); therefore, the last matrix will change to addition because the negative values are now within the matrix.

Matrix Multiplication

Let $[A]_{m\times n};\ [B]_{n\times p}$, be real-valued matrices with dimensions $m\times n$, & $n\times p$, respectively. Let $a_{i,j}\in[A],\ b_{i.j}\in[B]$, respectively, be any element in either matrix.

Then $[A]_{m\times n}\times[B]_{n\times p}=[C]_{m\times p}$, where $c_{i,j}\in[C]_{m\times p}$

In other words, when two matrices are multiplied, the number of columns in the first matrix must equal the number of rows in the second matrix. The product of this multiplication is a new matrix $[C]_{n\times p}$ with the same number of rows as $[A]_{m\times n}$ and same number of columns as $[B]_{n\times p}$.

Matrix Multiplication

If matrix $[A]$ has dimensions $m\times n$ and matrix $[B]$ has dimensions $n\times p$, then the product $[A]\cdot[B]=[C]$ has dimensions $m\times p$.

Without defining the actual process of matrix multiplication, follow the next examples to discern the general rule as to how the individual elements in a matrix are multiplied.

Matrix multiplication involves multiplication and addition. To multiply any two matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

Example #2: $\begin{bmatrix} 5 & 4 & 2 \end{bmatrix}_{1\times 3} \cdot \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}_{3\times 1} = \left[(5\cdot 6)+(4\cdot 1)+(2\cdot 3)\right] = \begin{bmatrix} 40 \end{bmatrix}_{1\times 1}$

Notice that a 1 × 3 matrix multiplied by a 3 × 1 matrix results in a 1 × 1 matrix. To multiply any two matrices, the *inner dimensions* must be the same, and then the *outer dimensions* become the dimensions of the resulting product matrix.

Outer dimensions

$[P]_{2\times 3} \quad \times \quad [Q]_{3\times 4} \ = 2 \ \times \ 4$ Product dimensions

Inner dimensions
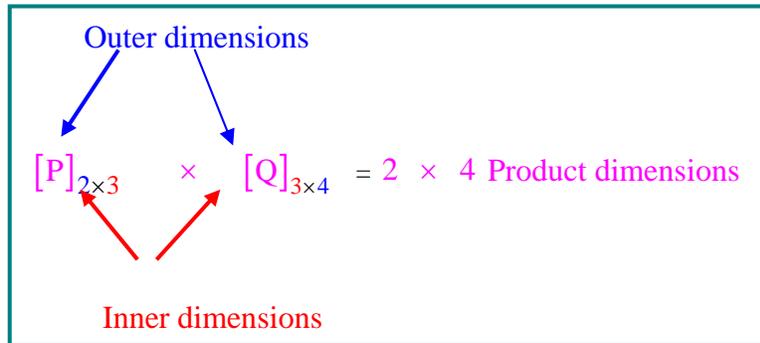
Example #3: Multiply:

$$\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}_{2\times 2} \cdot \begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}_{2\times 3} = \begin{bmatrix} (5\cdot 4)+(3\cdot 0) & (5\cdot 2)+(3\cdot 1) & (5\cdot -1)+(3\cdot 3) \\ (0\cdot 4)+(1\cdot 0) & (0\cdot 2)+(1\cdot 1) & (0\cdot -1)+(1\cdot 3) \end{bmatrix} = \begin{bmatrix} 20 & 13 & 4 \\ 0 & 1 & 3 \end{bmatrix}_{2\times 3}$$

* Notice that you take the first row, "5, 3" of the first matrix, and multiply that row by each column, successively, in the second matrix. You then move to the second row of the first matrix and multiply that row by each column in the second matrix, and so on.

Entering Matrices on the Graphing Calculator

The TI-83+ can be used to perform many matrix operations. However, as in previous units, it is often more useful to perform Matrix Algebra on paper prior to using the calculator in order to gain an intuitive knowledge for the operations involved.

Example #1:

Let: $[A]_{2\times3} = \begin{bmatrix} 2 & -4 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ $[B]_{3\times4} = \begin{bmatrix} 2 & -1 & -1 & 5 \\ 6 & 8 & 0 & 9 \\ 0 & -2 & 1 & 1 \end{bmatrix}$

$[C]_{3\times2} = \begin{bmatrix} 3 & 4 \\ -1 & 8 \\ 0 & -5 \end{bmatrix}$ $[D]_{2\times3} = \begin{bmatrix} 1 & 4 & 0 \\ -3 & -2 & 7 \end{bmatrix}$

Find the following using the graphing calculator:

a.) $[A]\cdot[B]$ b.) $[A]+[D]$ c.) $[D]\cdot[B]$ d.) $-4[C]$ e.) $[A]-[C]$

Step #1: Press $\boxed{2nd}$, $\boxed{x^{-1}}$ to access the calculator's Matrix Editor. Move cursor to the 'EDIT' menu and select #1 to edit matrix $[A]$. (Note: If matrix $[A]$ is not empty, this will not present a problem as we can simply type over the previous items).

Step #2: Set the dimensions of $[A]$ to '2×3'.

Step #3: Enter all values into $[A]$ as they appear above. Repeat this process for $[B], [C], \& [D]$.

Step #4: Return to the calculator's home screen by pressing, $\boxed{2nd}$, $\boxed{MODE}$

For Example #1, a: Find $[A]\cdot[B]$.

Press $\boxed{2nd}$, $\boxed{x^{-1}}$, $[A]$, $\boxed{ENTER}$
Press $\boxed{2nd}$, $\boxed{x^{-1}}$, $[B]$, $\boxed{ENTER}$, $\boxed{ENTER}$

Your calculator's screen should appear as:

```
[A][B]
[[-20  -34  -2  -2...
 [10   -7   -4  26...
```

If you use your right arrow, you can view the rest of the matrix.

```
[A][B]
...20  -34  -2  -26]
...0   -7   -4  26 ]]
```

Notice how the calculator uses the brackets, "[ ]", to show matrices.

For Example #1, b: Find: $[A]+[D]$

Select $[A]$ and $[D]$ according to the steps outlined above, but this time include a "+" sign between the two matrices.

Answer:

```
[A]+[D]
        [[3  0   0]
         [2  -2  8]]
```

Repeat this process for the remainder of the examples. Why do you think $[A]-[C]$ results in a calculator error?