SOLVING SYSTEMS WITH MATRIX EQUATIONS

In this unit you will explore inverse matrices and use them to solve systems of linear equations.

Square Matrix

The Identity Matrix for Multiplication

The Inverse of a Matrix

Determinant of a 2×2 Matrix

Solving Systems with Matrix Equations

Square Matrix

During World War II, Navaho code talkers, 29 members of the Navaho Nation, developed a code that was used by the United States Armed Forces. Matrices can be used to interpret secret codes.

A matrix can be used to encode a message and another matrix, **it's inverse**, is used to decode a message once it is received.

A **square matrix** is a matrix that has the same number of rows and columns.

Example #1: $\begin{bmatrix} 2 & 3 \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 4 & -9 & 5 \\ 7 & -3 & 8 \\ 0 & 2 & -1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} -2 & -8 & 4 & 8 \\ 0 & 1 & -3 & 6 \\ 4 & -4 & 0 & 9 \\ 11 & 32 & 57 & 3 \end{bmatrix}$

2×2 $\qquad$ 3×3 $\qquad$ 4×4

The Identity Matrix for Multiplication

Let A be a square matrix with n rows and n columns. Let I be a matrix with the same dimensions and with 1's on the main diagonal and 0's elsewhere. Then:

$$AI = IA = A.$$

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of a real number and 1 is the same number. The product of a square matrix B, and its identity I, is the matrix B.

Example #1: $$\underbrace{\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}}_{\boldsymbol{B}} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{I}} = \begin{bmatrix} (-4\cdot 1)+(4\cdot 0) & (-4\cdot 0)+(4\cdot 1) \\ (2\cdot 1)+(8\cdot 0) & (2\cdot 0)+(8\cdot 1) \end{bmatrix} = \underbrace{\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}}_{\boldsymbol{B}}$$

$\boldsymbol{B} \times \boldsymbol{I} = \boldsymbol{B}$

The Inverse of a Matrix

Let A be a square matrix with n rows and n columns. If there is an $n \times n$ matrix B, such that $AB = I$ *and* $BA = I$, then A and B are inverses of one another.

The inverse of matrix a is denoted by A^{-1}.

(Note: $A^{-1} \neq \frac{1}{A}$)

The product of a real number and its multiplicative inverse is 1. The product of a square matrix and its inverse is the identity matrix I.

Example #1: Let $C = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$

To show that C and D are inverses of one another multiply CD and DC.

$$CD = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (2\cdot -7)+(5\cdot 3) & (2\cdot 5)+(5\cdot -2) \\ (3\cdot -7)+(7\cdot 3) & (3\cdot 5)+(7\cdot -2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$DC = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} (-7\cdot 2)+(5\cdot 3) & (-7\cdot 5)+(5\cdot 7) \\ (3\cdot 2)+(-2\cdot 3) & (3\cdot 5)+(-2\cdot 7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since the product of these two matrices CD and DC are equal to the identity matrix they are inverses of each other.

Determinant of a 2 × 2 Matrix

Each square matrix can be assigned a real number called the ***determinant*** of the matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A, denoted by det(A) or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined as:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Matrix A has an inverse, if and only if, $\det(A) \neq 0$

Example #1: Find the determinant of matrix A, and then determine if matrix A has an inverse.

$$A = \begin{bmatrix} 7 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\det(A) = ad - bc$$
$$\det(A) = 7(-1) - 3(2)$$
$$\det(A) = -13$$

Since $\det(A) \neq 0$, matrix A has an inverse.

To use the determinant to find the inverse:

1.) find the difference of the cross products
2.) put this number under 1 and multiply it with the matrix using the following changes:
   a.) change the location of a and d in the matrix
   b.) change the signs of b and c in the matrix

Example #2: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

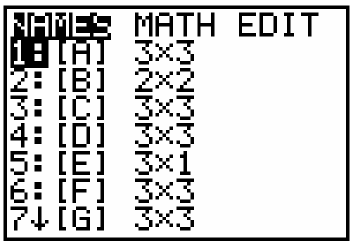
$$B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad B^{-1} = \frac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

To find the inverse of a 3 × 3 matrix, you will need to use your calculator.

1.) Press $\boxed{\text{2ND}}$, $\boxed{x^{-1}}$ to view the matrix features if you are using a TI-83Plus. (If you are using a TI-83 just push the button that says MATRIX.)



Don’t worry if there are numbers beside each of the matrices; we are going to be changing these.

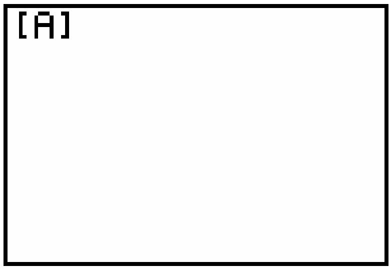
2.) Scroll over to EDIT and press enter. You will be entering data into matrix A.

3.) Change the dimensions to 3 × 3 by pressing 3 $\boxed{\text{ENTER}}$, 3 $\boxed{\text{ENTER}}$. Enter the numbers of your matrix into each position. Press $\boxed{\text{ENTER}}$ after each number.

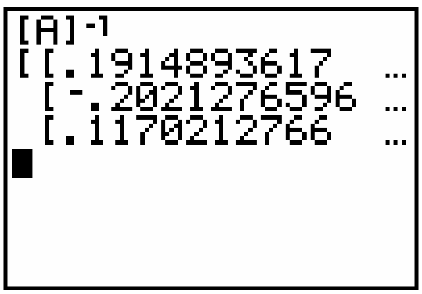
$$\begin{bmatrix} 12 & 7 & 1 \\ 3 & 4 & 2 \\ 5 & 3 & -3 \end{bmatrix}$$

4.) When you have entered the last number, press $\boxed{\text{2ND}}$, $\boxed{\text{MODE}}$ to go back to the home screen.

5.) To access matrix A, press $\boxed{\text{2ND}}$, $\boxed{x^{-1}}$ and then $\boxed{\text{ENTER}}$. $[A]$ should be on your screen.



6.) To find the inverse, press $\boxed{x^{-1}}$; your screen should have $[A]^{-1}$ showing. Press $\boxed{\text{ENTER}}$.



Notice that on your screen, you are unable to view the entire matrix. If you use the right arrow key, you can move across the screen to view the other columns.

$$A^{-1} = \begin{bmatrix} .19148... & -.25531... & -.10638... \\ -.20212... & .43617... & .22340... \\ .11702... & .01063... & -.28723... \end{bmatrix}$$

*To view all the entries, you may need to *scroll to the right*.

**To view the inverse as fractions, press $\boxed{\text{MATH}}$, $\boxed{\text{ENTER}}$, $\boxed{\text{ENTER}}$.

$$[A]^{-1} = \begin{bmatrix} \frac{9}{47} & \frac{-12}{47} & \frac{-5}{47} \\ \frac{-19}{94} & \frac{41}{94} & \frac{21}{94} \\ \frac{11}{94} & \frac{1}{94} & \frac{-27}{94} \end{bmatrix}$$

Solving Systems with Matrix Equations

A system of linear equations can be written as a matrix equation.

Example #1: $2a + 4b = -3$
$a - b = 9$

coefficient matrix, A	variable matrix, X	constant matrix, B
$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$	$X = \begin{bmatrix} a \\ b \end{bmatrix}$	$B = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

To solve:

1.) find the inverse of the **coefficient matrix**

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}, \qquad A^{-1} = \frac{1}{-2-4}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$$

2.) multiply both sides of the equation by A^{-1}

$$\overset{A^{-1}}{-\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}} \cdot \overset{A}{\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}} \cdot \overset{X}{\begin{bmatrix} a \\ b \end{bmatrix}} = \overset{A^{-1}}{-\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} -3 \\ 9 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} -33 \\ 21 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ -\frac{7}{2} \end{bmatrix}$$

$a = 5.5$ and $b = -3.5$