

TWO SPECIAL FUNCTIONS

Implied in our discussion of functions is the concept that a function may be continuous or discontinuous throughout its domain. Recall that in this function, $f(x) = 1/(x+2)$, a gap in the graph occurred at $x = -2$. This restriction in the function's domain was due to the fact division by zero is undefined mathematically. In essence, the laws of mathematics mandated this restriction. However, natural restrictions on the domain and/or range are only one type of mathematical function. If we intentionally restrict these values (or alternately expand them); we can create new functions and therefore new applications to various problem situations. In this unit we will examine special functions whereby restrictions on their domains and ranges lead to many real-world applications. These functions are called:

Piecewise Functions

Step Functions

Piecewise Function (Reference)

Floor [Greatest Integer] and Ceiling Functions (Reference)

Piecewise Functions

Example #1: Graph $f(x) = 2x - 1$ for $D_f = (-\infty, -2] \cup [2, \infty)$.

Normally $f(x) = 2x - 1$ is a linear function that is graphed over $D_f = (-\infty, \infty)$. For this example we have restricted the domain to **exclude** all x -values from $(-2, 2)$. We can view this new function on the graphing calculator.

Step #1: Press $\boxed{Y=}$ and type in the following: $Y_1 = (2X - 1)$. Be sure that the quantity $(2X - 1)$ is in $()$'s.

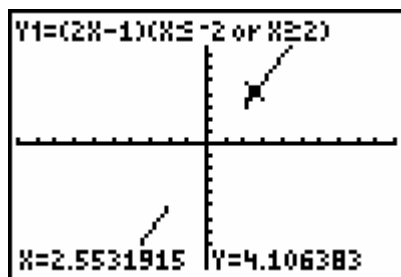
Step #2: With your cursor at the end of the last $)$, type in $(X \leq -2$ or $X \geq 2)$.*

*To find the inequality symbols, type $\boxed{2nd}$ \boxed{MATH} .

*To find the "or" command, type $\boxed{2nd}$ \boxed{MATH} , then right arrow to the 'LOGIC' menu.

" $Y_1 =$ " should now display: $Y_1 = (2X - 1) (X \leq -2$ or $X \geq 2)$

Step #3: Press \boxed{GRAPH} or \boxed{ZOOM} 6.



The function is now graphed as two separate lines.

This simple example illustrates the concept that, 'a function is entirely determined by its domain and range values' and new functions can be 'created' by restricting (or expanding) these values. These 'created' functions are called "Piecewise Functions".

Sample Applications of Piecewise Functions

In order that we may understand the importance of piecewise functions and their applications consider the following problem:

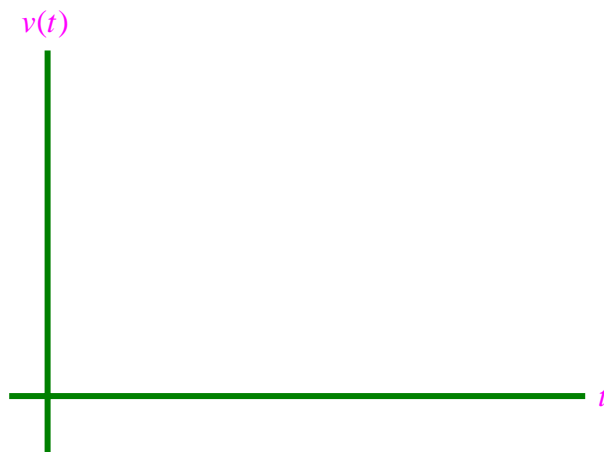
Sample #1: An airplane leaves Cleveland in route to Chicago. The plane accelerates for 20 minutes until it reaches a cruising speed of 450 mph and travels at this speed for 45 minutes. As the plane approaches Chicago, it begins to slow and then lands 15 minutes later. Describe the plane's flight graphically.

The plane's flight is described in *three* stages:

- (1) The plane takes off and accelerates for 20 minutes.
- (2) The plane travels at a constant rate for 45 minutes.
- (3) The plane slows and comes to a stop 15 minutes later.

Each part of the plane's flight describes a separate 'piece' of its behavior. When the flight is graphed, there should be three distinct sections of the graph to view and interpret.

In the problem we identify two variables in the plane's flight: the plane's velocity (v) and the amount of time the plane travels at that velocity (t). These two variables can be used to replace the x - and y -axes on the xy -plane in the following manner:



*For our graph the x -axis represents time (t) and the y -axis is labeled in functional notation for velocity ($v(t)$).

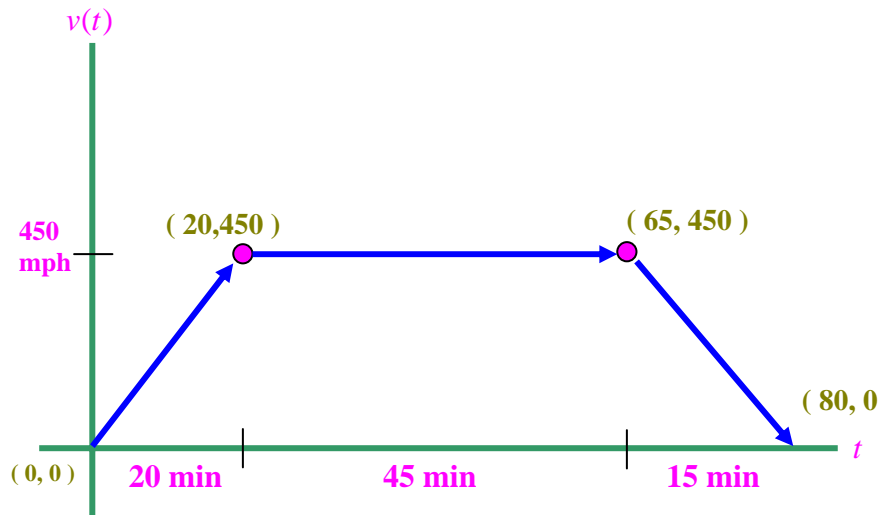
From your science classes you may recall that an object's velocity is defined to be the distance traveled for a given amount of time or $v = \frac{d}{t}$. You will also recall that an object's acceleration is defined to be the change in velocity over a given period of time or $a = \frac{v}{t}$. In both formulas, the object's velocity and acceleration depend on time.

In math we say that velocity and acceleration are both 'functions' of time. Both can be expressed in functional notation as:

Velocity: $v(t) = \frac{d}{t}$

Acceleration: $a(t) = \frac{v}{t}$

With our graph appropriately labeled we can describe the plane’s flight accordingly. (We will assume that the plane travels uniformly in all three situations with no delays, wind interference, waiting to taxi, etc.).



For this graph we make note of the following:

- (1) The total amount of time the plane spends traveling is 85 minutes.
- (2) The plane obtains a maximum speed of 450 mph, 20 minutes into the flight, and maintains this speed for the next 45 minutes.
- (3) Sixty-five minutes after the plane takes off, it begins to slow for 15 minutes until it comes to rest in Chicago.

Each section of the graph is a line segment. We can find the equations of these lines which we can use to discern more information about the plane’s flight.

In order to find a mathematical representation for each ‘piece’ of the plane’s flight we need to recall our linear equation formulas.

Slope Formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

Point –Slope Formula: $y - y_1 = m(x - x_1)$

For our investigation we will transform the above formulas to represent the variables on our graph. Substituting t - values for x and $v(t)$ - values for y we obtain the following:

$$\text{Slope Formula: } m = \frac{y_2 - y_1}{x_2 - x_1}, \Rightarrow a = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$\text{Point -Slope Formula: } y - y_1 = m(x - x_1), \Rightarrow v(t) - v(t_1) = a(t - t_1)$$

We can use these formulas to find equations for each section of the graph. First we will establish domain and range values for each piece since this is a **restricted function**.

$$\text{For piece \#1: } D_v = [0, 20]; \quad R_v = [0, 450]$$

$$\text{For piece \#2: } D_v = [20, 65]; \quad R_v = [450]$$

(note: the range for piece #2 has constant value).

$$\text{For piece \#3: } D_v = [65, 80]; \quad R_v = [450, 0]$$

Next we will use the endpoints of each piece to develop mathematical equations for each piece using our modified linear equations.

$$\text{For piece \#1: } a = \frac{450 - 0}{20 - 0} = 22.5 \text{ and } v(t) - 0 = 22.5(t - 0) \Rightarrow v(t) = 22.5t$$

$$\text{For piece \#2: } a = \frac{450 - 450}{65 - 20} = 0 \text{ and } v(t) = 450$$

$$\text{For piece \#3: } a = \frac{0 - 450}{80 - 65} = -30 \text{ and } v(t) - 0 = -30(t - 80) \Rightarrow v(t) = -30t + 2400$$

We now have an equation that describes the plane's flight for each part of its journey. Each piece depends on the amount of time that the plane's flight followed that equation.

In science we might say that the flight is described by the 'parameter' of time. In mathematics the time parameter translates to a restriction on each equation's domain.

We can write each equation and their domain restrictions as one 'piecewise function' in the following manner.

$$v(t) = \begin{cases} 22.5t & \text{for } 0 \leq t < 20 & [0, 20) \\ 450 & \text{for } 20 \leq t < 65 & [20, 65) \\ -30t + 2400 & \text{for } 65 \leq t \leq 80 & [65, 80] \end{cases}$$

The above description is the common form to represent a piecewise function. It is important to note a few of its features:

The one sided '{' indicates that a piecewise function is a combination of multiple sets.

The use of the half-open interval around one point in one piece, and the closed interval notation in the next piece, is to indicate that one endpoint of an interval may not exist on two pieces of the function simultaneously. However where we place a '[' or a '(' may be completely arbitrary provided the selection is consistent.

For example, the domain could have been stated as:

$$0 \leq t \leq 20 \quad [0,20]$$

$$20 < t < 65 \quad (20,65)$$

$$65 \leq t \leq 80 \quad [65,80]$$

For piecewise functions that contains gaps, the selection of '[']' or '()' may not be arbitrary.

Using both inequalities and interval notation is not necessary. In this course both may be used.

Example #2: Graph the above problem on the TI-83+ as a piecewise function.

The graphing calculator will graph a set of “connected” piecewise functions as one graph according to the domain restrictions and using the symbols of inequality as outlined at the beginning of this unit. Follow similar procedures to these when you are asked to graph a piecewise function on the calculator.

Step #1: Set the calculator’s window settings according to the domain and range values indicated above.

$$X_{\min} = -1; \quad X_{\max} = 85$$

$$Y_{\min} = -1; \quad Y_{\max} = 510$$

Step #2: Type each piece into $\boxed{Y=}$ in ()’s, followed by the restrictions. Your equations should look like the following.

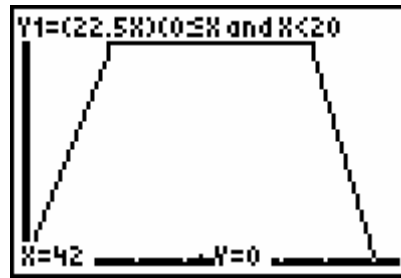
*Remember to use [2nd][MATH] to access the inequality symbols and the “LOGIC” commands.

$$Y_1 = (22.5X)(0 \leq X \text{ and } X < 20)$$

$$Y_2 = (450)(20 \leq X \text{ and } X < 65)$$

$$Y_3 = (-30X + 2400)(65 \leq X \text{ and } X \leq 80)$$

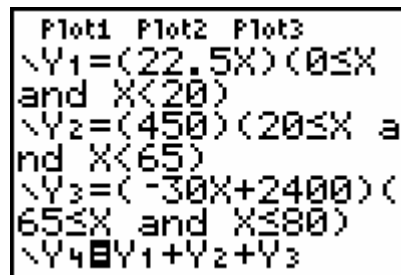
Step #3: Press **GRAPH**



Another way to graph the above piecewise function is:

Step #4: Press **VAR**, **Y-VARS**, **ENTER**, to find the commands to be entered into **Y4=**. Enter $Y_1 + Y_2 + Y_3$

Step #5: Turn the equals signs off by highlighting each one, then press **ENTER**. The following should be in **Y=**, then **GRAPH**. (Did this work?).



Example #3: Graph the following piecewise function.

$$f(x) = \begin{cases} x^2 - 4 & \text{for } [-5, -3] \\ 3x + 7 & \text{for } (0, 6] \\ -2x^2 & \text{for } [8, 11] \end{cases}$$

Step #1: For each piece of this function, the endpoints of the intervals can provide us with an idea of the range of the function. This is useful so that we can scale our graph paper or calculator's window settings accordingly.

Find $f(-5)$ and $f(-3)$ for $f(x) = x^2 - 4$

$$f(-5) = 21$$

$$f(-3) = 5$$

Find $f(0)$ and $f(6)$ for $f(x) = 3x + 7$

$$f(0) = 7$$

$$f(6) = 25$$

*Even though $x = 0$ is not on the domain of this piece, we still calculate $f(0) = 7$ to help scale the graph.

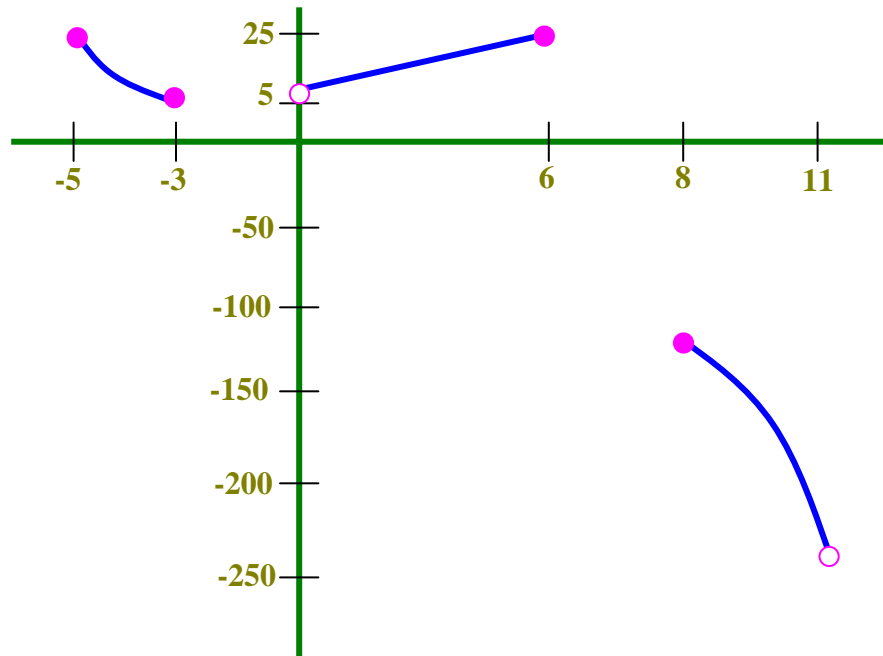
Find $f(8)$ and $f(11)$ for $f(x) = -2x^2$

$$f(8) = -128$$

$$f(11) = -242$$

If we assume that each section of this function does not take on even more extreme values between the endpoints than the ones calculated, then we see that our graph is scaled over a wide range of values. (Calculating extreme values between endpoints is left for a later unit).

To graph the function we set up the following axes:



The above graph represents a piecewise function with multiple gaps both vertically and horizontally. The first piece is a continuous section of a parabola on a closed interval. The second piece is the equation of a line on a half-open interval. The third section is also a section of a parabola on a half-open interval.

This function could not be easily graphed on the calculator. First, the extreme window settings would cause the graph to be distorted. Secondly, the calculator may connect the graph to the x -axis at multiple points. And lastly, the calculator would not graph 'half-open' intervals as open circles.

If you would like to attempt this function on your calculator on your own, enter the following with appropriate window settings:

$$Y_1 = (X^2 - 4)(-5 \leq X \text{ and } X \leq -3)$$

$$Y_2 = (3X + 7)(0 < X \text{ and } X \leq 6)$$

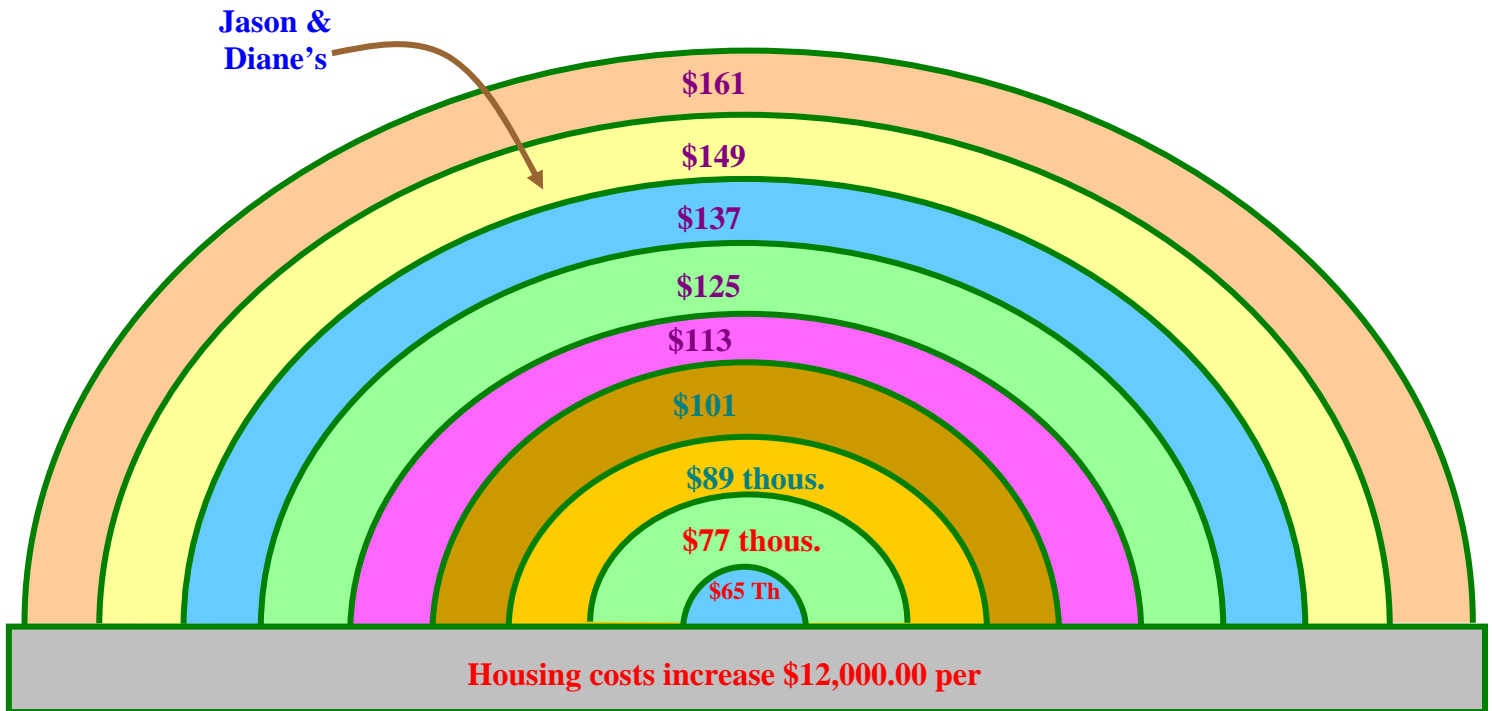
$$Y_3 = (-2X^2)(8 \leq X \text{ and } X < 11)$$

Step Functions

Consider the following situation:

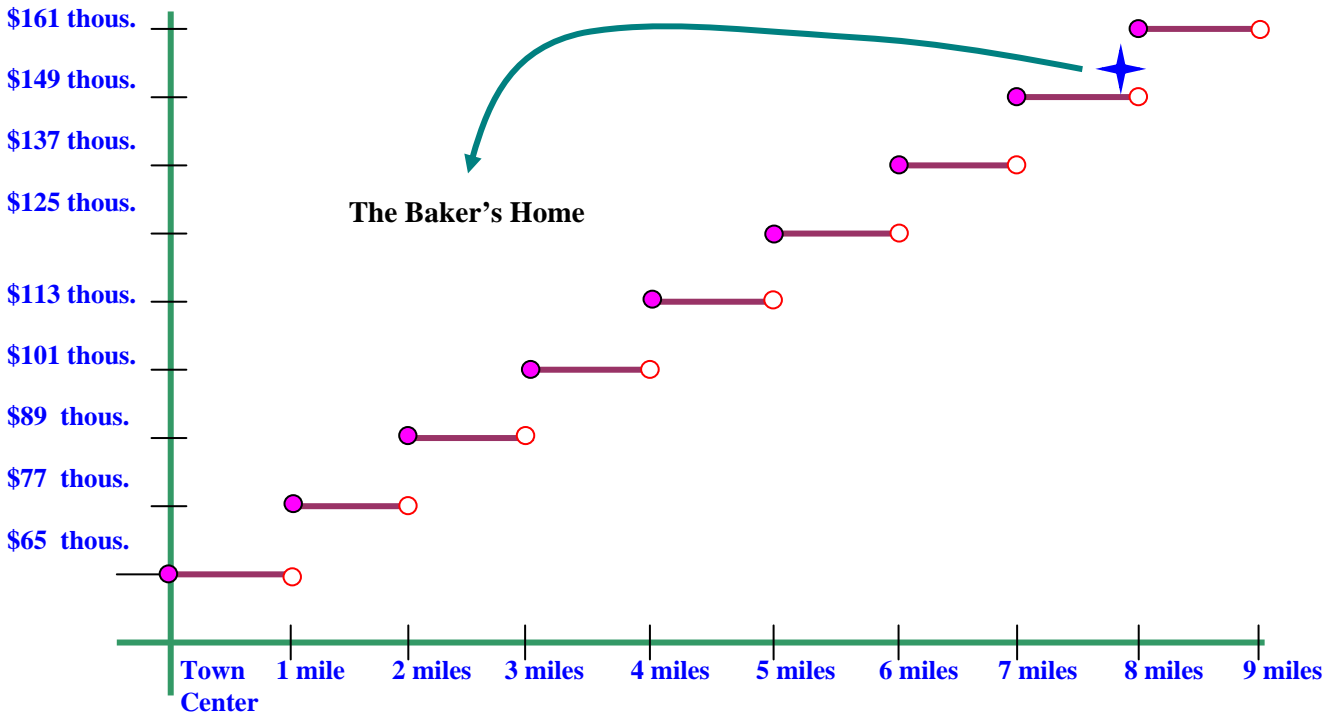
Jason and Diane Baker are newly married and are looking to buy a home in their town. Jason works downtown as a computer consultant. Diane works 17 miles from downtown as an architect. Jason and Diane can afford a home from between \$130,000.00 to \$150,000. Jason and Diane want to buy a house that is roughly equal driving distance for both and still affordable in their price range. As the Baker's examine housing prices, they notice housing costs are a constant \$65,000.00 from the center of town to one mile out of town. Between one and two miles, homes jump by \$12,000.00 to \$77,000.00. Between two and three miles, prices jump to \$89,000.00. This \$12,000.00 increase continues every mile from the center of town. What is the maximum distance that Jason and Diane will drive one way to work each day?

One way to answer this problem is to plot housing costs increases from the center of town as a series of concentric circles with the radius of the innermost circle equal to one mile and each circle increasing in radius by one mile.



From this we see; if Jason and Diane live at the outermost edge of the 8 mile circle, then Jason will drive just under 8 miles to work and Diane will drive just over 9 miles –roughly equal driving for both.

Another way to display this situation is to plot the graph on the xy -plane where the x -axis equals the distance from the center of town and the y -axis represents the housing costs at that distance.



As you can see in the above graph, housing costs remain the same every mile but jump to the next level at the beginning of each new mile. If a person had \$85,000.00 to spend on a home and wanted to live as far away from downtown as possible, then –as the graph indicates- he or she could live no more than 2 miles from the center of town. At less than 2 miles housing costs are constant at \$77,000.00, then jump to \$89,000.00 which is beyond an \$85,000.00 price range.

Ceiling Function

Let $f(x) \in \mathbb{R}$ be a real-valued function. Also define (a, b) to be an open interval from a to b where $z \in (a, b)$ is the only integer value on the interval. Therefore $a < z < b$ and all other numbers on (a, b) have decimal value not equal to a whole number.

Then $f(x) = \lceil x \rceil$ is called, “The Ceiling Function” of x and is defined as a piecewise function of the following form:

$$f(x) = \llbracket x \rrbracket = \begin{cases} z, & \text{if } a < x \leq z \\ z+1, & \text{if } z < x < b \end{cases}$$

In order to see how this function operates, we choose $f(x) = \llbracket x \rrbracket$ under the following conditions:

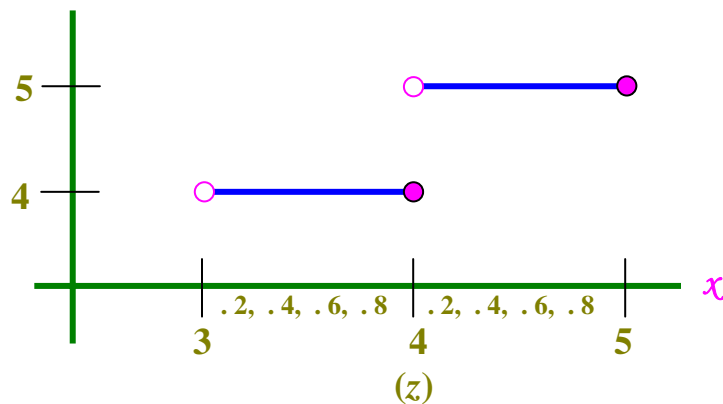
Let: $a = 3.1, b = 4.9 \Rightarrow (a, b) = (3.1, 4.9)$ and $4 \in (3.1, 4.9)$.

In a table of values select a few numbers from $(a, b) = (3.1, 4.9)$ and $z = 4 \in (3.1, 4.9)$

X	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8
$\llbracket x \rrbracket$	4	4	4	4	4	5	5	5	5

As noted in the initial conditions for this problem, $z = 4$ (the integer value on the interval $(3.1, 4.9)$). From our piecewise description of this function, 4 will be the function value recorded in our table if we choose an x (a domain value) that is less than 4. If we choose 4, then the function value is still 4. But if we select a domain value larger than 4, then our range jumps to 5. This is illustrated on the following graph:

$$Y = f(x) = \llbracket x \rrbracket$$



If we continue the above example into the next interval for $(a, b) = (4.1, 5.9)$, then $z = 5$ and the function's graph jumps to the next level and so on.

A step function has many real world applications. Doctors use a chart of greatest integers to know how much prescription to give to their patients based on body weight. If a patient weighs 145 lbs. then he/she might receive a dosage of 10 milligrams of a medicine because that patient falls between 140-150 lbs. However a patient who weighs 155 lbs. might receive 15 milligrams because that patient has 'stepped up' to the next 10 lb. weight category.

Floor Function (Greatest Integer Function)

The floor function $f(x) = \llbracket x \rrbracket$, also called the greatest integer function, gives the largest integer value less than or equal to x .

Graph the following Floor function over $D_f = (-2, 2.9)$.

$f(x) = \llbracket x \rrbracket$: For this problem there are 5 z -values over $(-2.9, 2.9)$. They are: $\{-2, -1, 0, 1, 2\}$. If we separate the interval $(-2, 2.9)$ so that only one z -value is on an interval (or sub-interval) at a time, then we might examine $D_f = (-2, 2.9)$ separated in the following manner:

First sub-interval: $(a, b) = (-2, -1.1)$	$z_1 = -2$
Next sub-interval: $(a, b) = (-1, -0.1)$	$z_2 = -1$
Next sub-interval: $(a, b) = (0, 0.9)$	$z_3 = 0$
Next sub-interval: $(a, b) = (1, 1.9)$	$z_4 = 1$
Final sub-interval: $(a, b) = (2, 2.9)$	$z_5 = 2$

Tables of values for two of the subintervals are given below. The remaining sub-intervals can be calculated on your own.

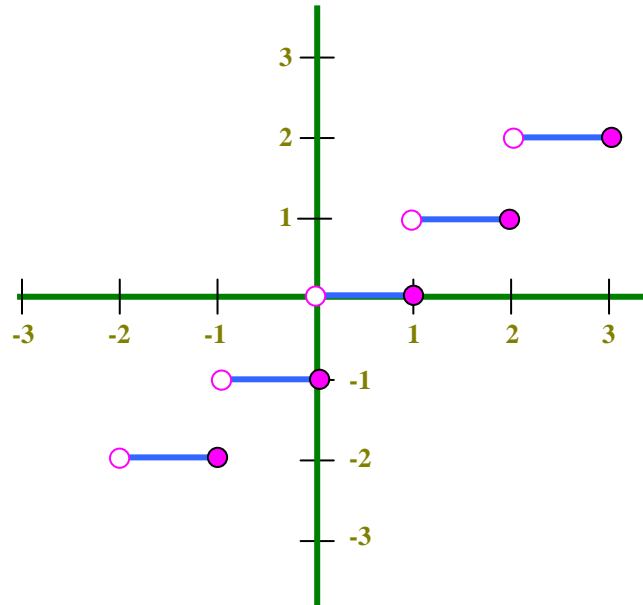
Table for $z = -2$:

X	-2	-1.9	-1.8	-1.6	-1.4	-1.3	-1.2	-1.1
$\llbracket x \rrbracket$	-2	-2	-2	-2	-2	-2	-2	-2

Table for $z = 1$:

X	1.0	1.1	1.3	1.4	1.5	1.7	1.8	1.9
$\llbracket x \rrbracket$	1	1	1	1	1	1	1	1

The resulting graph would appear as the following:



Sample Problem

$f(x) = 2[x]$: Again, there are 5 z -values over $(-2.9, 2.9)$. They are: $\{-2, -1, 0, 1, 2\}$. We will again separate the interval $(-2, 2.9)$ so that only one z -value is on any sub-interval at a time, then we see $D_f = (-2, 2.9)$ again separated in the following manner:

First sub-interval: $(a, b) = (-2, -1.1)$	$z_1 = -2$
Next sub-interval: $(a, b) = (-1, -0.1)$	$z_2 = -1$
Next sub-interval: $(a, b) = (0, 0.9)$	$z_3 = 0$
Next sub-interval: $(a, b) = (1, 1.9)$	$z_4 = 1$
Final sub-interval: $(a, b) = (2, 2.9)$	$z_5 = 2$

Tables of values for the first two subintervals are given below. The remaining sub-intervals can be calculated on your own. This time, however, after we evaluate $[x]$, we then multiply the result by 2. (Be sure to find $[x]$, then multiply. This proceeds just as ‘order of operations’ would).

Table for $z = -2$:

X	-2	-1.9	-1.8	-1.6	-1.4	-1.3	-1.2	-1.1
$\llbracket x \rrbracket$	-4	-4	-4	-4	-4	-4	-4	-4

Table for $z = -1$:

X	-1.0	-0.9	-0.7	-0.5	-0.4	-0.3	-0.2	-0.1
$\llbracket x \rrbracket$	-2	-2	-2	-2	-2	-2	-2	-2

The resulting graph would appear as the following: Notice that every jump is twice as high as the previous graph.

