THE INVERSE OF A FUNCTION

In this unit we examine the concept of the Inverse of a Function, the meaning of a “One-to-One” function, and continue our analysis of these concepts on the graphing calculator.

The Inverse of a Function

Recall that the composition of functions was defined as the following:
Let $f(x),\ g(x) \in$ be real-valued functions and denoted by $f = f(x),\ g = g(x)$, then the Composition of Functions is: $f \circ g = f(g(x))$.

For many applications the composition of functions is insignificant. However, when the composition results in a special situation, then we make particular note and define the situation accordingly:

The Inverse of a Function:

Let $f = f(x), g = g(x) \in \mathbb{R}$ be real-valued functions.
If $f(g(x)) = g(f(x)) = x$, we say that f and g are Inverse Functions.
The Inverse of a function f is denoted by: $f^{-1} = f^{-1}(x) = g(x)$

Example #1: Determine if $f(x) = x + 4$ and $g(x) = x - 4$ are inverse functions.
From the above definition it is necessary that f and g satisfy two conditions:

(1.) Find: $f(g(x)) = f(x-4) = (x-4)+4 = x$

(2.) Find: $g(f(x)) = g(x+4) = (x+4)-4 = x$

Therefore $f(x) = x + 4$ and $f^{-1}(x) = x - 4$.

Example #2: Determine if $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions.

(1.) Find: $f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

(2.) Find: $g(f(x)) = g = \sqrt{(x^2)} = x$ or $-x$

Since $g(f(x))$ returns two possible values, we cannot say that f and g are inverses. Recall from the previous unit, however, that by restricting the domain and/or range of g we were able to analyze $g(x)$ as a restricted function. If we restrict one of these values, then we may also find that $g(x)$ is a restricted inverse of $f(x)$. We will explore this possibility shortly.

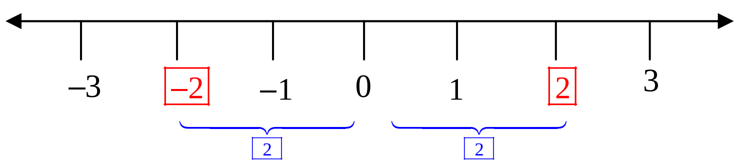
Graphical Representation of the Inverse

In your previous math courses you may have learned that pairs of numbers such as -8 and 8, -21.3 and 21.3, and etc. are called "Additive Inverses" and that numbers such as $\frac{4}{7}$ and $\frac{7}{4}$, and -5 and $\frac{-1}{5}$, are called "Multiplicative Inverses". The reason that these pairs of numbers are significant is that, when added or multiplied, the result is either "0" (for the additive inverses) or "1" (for the multiplicative inverses).

$$(\text{ i.e.} -8+8=0 \text{ and } \frac{7}{4}\times\frac{4}{7}=1\ )$$

Additive Inverse

Graphically the additive inverses are illustrative of the significance of the inverse concept to mathematics.



Both -2 and 2 are exactly 2 units (or the same distance) from zero on the number line. This is true of all sets of additive inverses. We therefore view the number line as being "symmetric" about 0.

Multiplicative Inverse

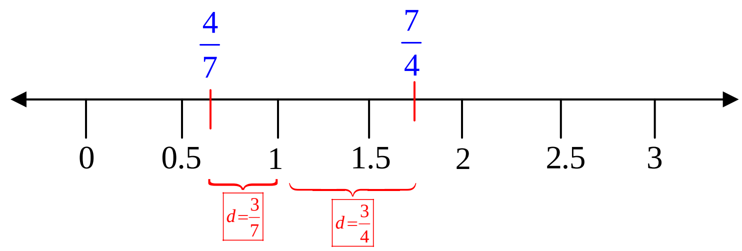
For multiplication the inverses represent a different form of symmetry about the number one. From your study of Geometry you may recall the proportional value of the Geometric Mean. Recall that the following proportion defines the geometric mean:

$$\frac{a}{b}=\frac{b}{c} \quad \text{or} \quad b^2=ac$$

For the number line the number, 1, is the geometric mean between any two multiplicative inverses.

$$(\text{ i.e. } \frac{\frac{4}{7}}{1}=\frac{1}{\frac{7}{4}} \quad \Rightarrow \quad 1^2=\frac{4}{7}\times\frac{7}{4}=1\ )$$

Graphically the geometric mean is represented as:



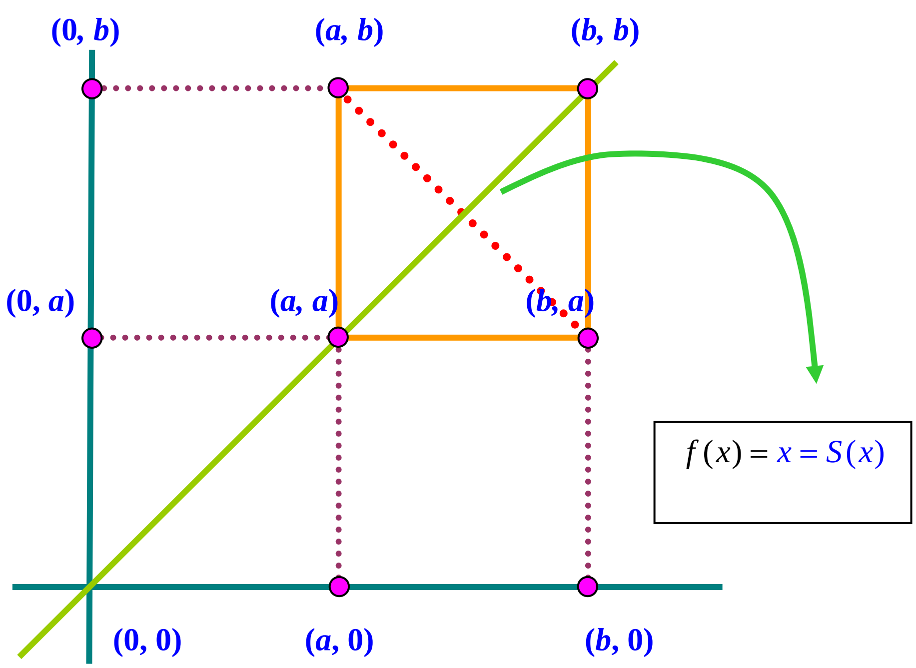
Although each number's distance from one is not equal, you will notice that their numerators are equal. This indicates that the multiplicative inverses are proportionally symmetric about the number one. (The graphical representation for negative multiplicative inverses is similar to that shown for positive inverses and can be investigated on your own).

The key concept in the study of inverses is symmetry about a particular value. Since functions are relations between two sets and represented as ordered pairs, we need to find a value about which each point in the set is symmetrical.

The Inverse of a Coordinate Point:

Let $a,\ b \in \mathbb{R}$ and $a \le b$ and (a,b) be an ordered pair on the xy-plane. Then the ordered pair (b,a) is the inverse point of (a,b).

Graphically (for illustrative purposes we will show $a \ne b$ although by definition $a \le b$):



Note that points (a,a), and (b,b) each define a square with

$(0,0)$, $(a,0)$, and $(0,a)$ as the three other vertices with (a,a) and

$(0,0)$, $(b,0)$, and $(0,b)$ as the three other vertices with (b,b).

In addition the points

(a,a), (a,b), (b,b), and (b,a) also define a square with (a,a), (b,b) along one diagonal and (a,b), (b,a) along the other diagonal.

If we were to plot every point and its inverse on the xy-plane, we would find that an infinite number of such squares would be formed (for negative coordinates as well – investigate this on your own).

If we then connect all the diagonals of these squares, starting at vertex, $(0,0)$, we obtain a line whose equation is $y = x$ or $f(x) = x$, with $D_x = (-\infty, \infty)$ and $R_y = (-\infty, \infty)$.

Therefore we will say that the function is the "**Line of Symmetry"** for all points and their inverses on the xy-plane.

Recall again our definition of the Inverse of a Function:

"If $f(g(x)) = g(f(x)) = x$, we say that f and g are Inverse Functions".
We can now conclude that a function and its inverse are symmetrical about the line $y = x$.

Note: In our previous graph we also observe that the points
$(a,a), (a,b), (b,b),$ and (b,a) define a square with (a,a) and (b,b) forming one diagonal, and the inverse points (a,b) and (b,a) forming the other.

Recall from geometry that a square is a rhombus, and that the diagonals of a rhombus are perpendicular $(\perp)$ and bisect one another. Therefore the diagonal formed by connecting (a,b) and (b,a) is $\perp$ to the line of symmetry, $S(x) = x,$ and is bisected by this line.

Finding the Inverse of a Function

From our previous discussion on inverse points we notice that inverse points are found by interchanging (a,b) with (b,a). This indicates that inverse functions could be found by interchanging every point (x,y) on the function with the point (y,x). Algebraically this implies writing (or solving) a function in terms of its other variable.

Example #1: Given $f(x)=3x+7$, find $f^{-1}(x)$ and graph both on the same coordinate axes.

Step #1: Recall that $y=f(x)$, so we can rewrite our function as: $y=3x+7$.

Step #2: Solve for x:

$$y=3x+7$$

$$y-7=3x$$

$$\frac{y-7}{3}=x$$

Step #3: Interchange x and y:

$$y=\frac{x-7}{3}=g(x)$$

Step #4: Verify that $g(x)=f^{-1}(x)$

(1.) Find: $f(g(x))=f\left(\frac{x-7}{3}\right)=3\left(\frac{x-7}{3}\right)+7=(x-7)+7=x$

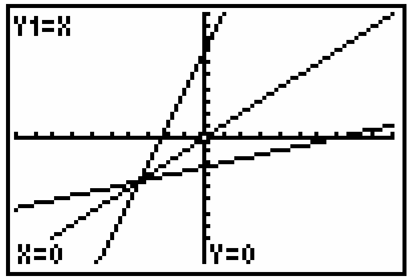
(2.) Find: $g(f(x))=g(3x+7)=\frac{(3x+7)-7}{3}=\frac{3x}{3}=x$

Therefore $f(x)=3x+7$ and $f^{-1}(x)=\frac{x-7}{3}$.

Step #5: Graph each equation on the graphing calculator.

Press [Y=] and enter the following, then press [GRAPH] or [ZOOM] 6:ZStandard.

$$Y_1 = X$$
$$Y_2 = 3X + 7$$
$$Y_3 = (X - 7)/3$$



Example #2: For $f(x) = x^2 - 6x + 8$, find $f^{-1}(x)$ and graph.

Step #1: Recall that $y = f(x)$, so we can rewrite our function as: $y = x^2 - 6x + 8$.

Step #2: Solve for x. Recall from Algebra II that in order to solve the above equation for x, we use the method of, "Completing the Square".

$$x^2 - 6x + 8 = y$$
$$x^2 - 6x \quad = y - 8$$
$$x^2 - 6x + 9 = y - 8 + 9$$
$$(x-3)^2 = (y+1)$$
$$x - 3 = \pm\sqrt{(y+1)}$$
$$x = 3 \pm \sqrt{(y+1)}$$

Step #3: Interchange x and y $\quad y = 3 \pm \sqrt{(x+1)} = g(x)$

Recall that an equation of this form, $(\pm\sqrt{\ })$, must be graphed as a restricted function. For our purposes, we will choose, $y = 3 + \sqrt{(x+1)} = g(x)$.

Step #4: Verify that $g(x) = f^{-1}(x)$

(1.) find: $f(g(x))$

$$= f(3+\sqrt{x+1}) = (3+\sqrt{x+1})^2 - 6(3+\sqrt{x+1}) + 8$$

$$= 9 + 6\sqrt{x+1} + x + 1 - 18 - 6\sqrt{x+1} + 8 = x$$

Note: Although we will review how to FOIL radical quantities later, students should possess a familiarity with this process from Algebra II.

(2.) find: $g(f(x))$

$$= g(x^2 - 6x + 8) = 3 + \sqrt{(x^2 - 6x + 8) + 1} = 3 + \sqrt{x^2 - 6x + 9}$$

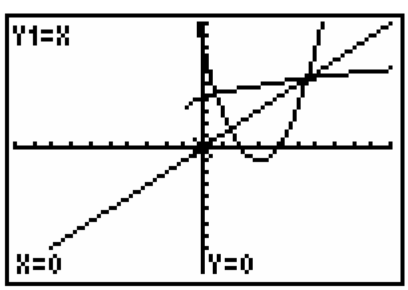
$$= 3 + \sqrt{(x-3)^2} = 3 + (x-3) = x$$

Note: Students should also possess knowledge on how to factor "trinomial squares".

Therefore $f(x) = x^2 - 6x + 8$ and $f^{-1}(x) = 3 + \sqrt{(x+1)}$ (under restrictions)

Step #5: Graph each equation on the graphing calculator. Enter the following into [Y=], then graph:

$$Y_1 = X$$
$$Y_2 = X^2 - 6X + 8$$
$$Y_3 = 3 + \sqrt{(X+1)}$$



Notice that $f(x)$ and $f^{-1}(x)$ intersect at two points and that one of those two points is on the line of symmetry. If all points of intersection between $f(x)$ and $f^{-1}(x)$ occur on the line of symmetry, the $f(x)$ and $f^{-1}(x)$ have a special relationship which we will examine next.

"One-to-One" Functions

As noted in our previous example, "Finding Inverse of a Function" link, at least one point of intersection between $f(x) = x^2 - 6x + 8$ and $f^{-1}(x) = 3 + \sqrt{(x+1)}$ occurs on the line of symmetry, $y = x$. However, another intersection does not. This can be one way to determine that the two functions are not 'one-to-one' (although one-to-one functions can have intersection points off the line, $y = x$). This graphical interpretation is only one aspect of what it means for a function to be a 'one-to-one' function.

One-to-One Functions

Let $f = f(x) \in \mathbb{R}$ be a real valued function and $f^{-1} = f^{-1}(x)$ be the inverse of f.

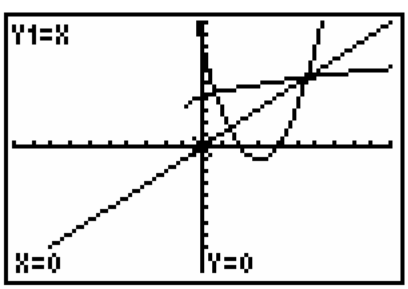
f is a **'one-to-one'** function if for every $x \in D_f$ and every $y \in R_f$, $y \in D_{f^{-1}}$ and $x \in R_{f^{-1}}$.

In other words, if all of the domain of f is the range of f^{-1}, and of the range of f is the domain of f^{-1}, then the two functions are **'one-to-one'** (denoted, **"**1-1**")**.

The fact that the graph in the second example in the link to "Finding Inverse of a Function" (shown below) was not 1-1 can now be explained from the definition.

$f(x) = x^2 - 6x + 8$ and $f^{-1}(x) = 3 + \sqrt{(x+1)}$ (under restrictions)

$$Y_1 = X$$
$$Y_2 = X^2 - 6X + 8$$
$$Y_3 = 3 + \sqrt{(X+1)}$$



For $f(x) = x^2 - 6x - 8$, we have $D_x = (-\infty, \infty)$ and $R_y = [-1, \infty)$. However, for $f^{-1}(x) = 3 + \sqrt{(x+1)}$, we have $D_{f^{-1}} = [-1, \infty)$ and $R_{f^{-1}} = [3, \infty) \neq D_f$

(Note: These Domain and Range values can be found using techniques from a previous unit.)

However if we restrict D_f , we can form a restricted, 1-1 function for $f(x)$. Let $D_f = [3, \infty)$. We now graph the two functions in the following way on the calculator.

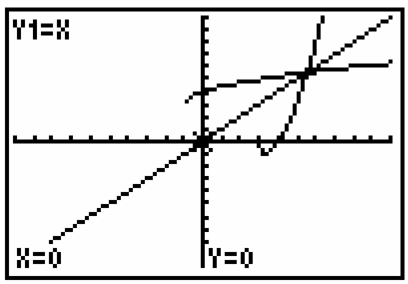
Example #1: To have the TI-83+ graph restrictions do the following: In [Y=] change Y_2 by typing:

$Y_1 = X$

$Y_2 = (X^2 - 6X + 8)(3 \le X)$*

$Y_3 = 3 + \sqrt{(X+1)}$

*The ' $\le$ ' command is found by pressing, [2nd] , [MATH]



Example #2: Let $f(x) = \dfrac{1}{x+2}$

(1.) State: D_f and R_f

(2.) Find: $f^{-1}(x)$

(3.) Determine if $f(x)$ is 1-1. If $f(x)$ is not 1-1**,** restrict either or both the D_f and R_f to make $f(x)$,1-1.

(4.) Graph both $f(x)$ and $f^{-1}(x)$ on the same display along with $y = x$ (the line of symmetry).

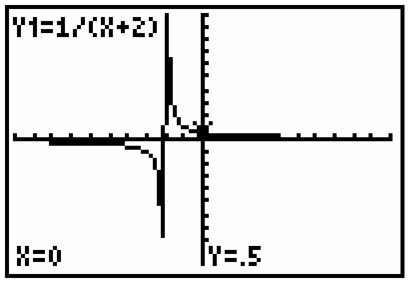
Part #1: State: D_f and R_f .

(1.) Clearly, $x \ne -2$ as this would cause the denominator to equal zero. Therefore $x = -2$ is restricted from the domain of $f(x)$. All other real numbers can however be used as a value for x in the denominator without causing problems.

Conclusion: $D_f = (-\infty,\ -2) \cup (-2,\ \infty)$

(2.) To find the range we will examine this function on the graphing calculator.

Clear other graphs and then enter $Y_1 = 1/(x+2)$, then press [GRAPH]
or [ZOOM] 6



The calculator will not display the entire graph and may appear to connect the curve at $x = -2$, which we know, is impossible.

Press [TRACE] then type "-2" [ENTER]. At the bottom of the display the calculator displays "-2" for x, but there is no y-value given. In addition, your tracing cursor is no longer on the graph. Although the calculator may not graph this function properly, it does 'know' that the function is undefined at $x = -2$.

Press, [TRACE] again and use your right and left arrows to find your cursor. Type the following numbers in order, pressing [ENTER] after each one. Record or make note of the y-values that result from each new number.

Type -2.001	$y =$?
Type -2.0001	$y =$?
Type -2.00001	$y =$?
Type -2.000001	$y =$?

What do you notice about your y-values as you allow x to come close to -2 from the left of -2?

Now type the following numbers in order and press ENTER after each one. Record or make note of the y-values that result from each new number.

Type -1.99	$y =$?
Type -1.999	$y =$?
Type -1.9999	$y =$?
Type -1.99999	$y =$?

What do you notice about your y-values as you allow x to come closer to -2 from the right of -2?

What is not clear about the calculator's graph, but what is clear about these numerical results, is that at $x = -2$, the graph of $f(x) = \dfrac{1}{x+2}$ increases to ∞ on one side of $x = -2$ and decreases to $-\infty$ on the other side. In addition, if we examine the graph around the x- axis we see that to the right of $x = -2$, the function values always remain above the axis, while to the left of $x = -2$, the function values are always negative and below the axis. This occurs because the expression $\dfrac{1}{x+2} > 0$ for all $x > -2$, and the expression is also < 0 for all $x < -2$ (test a few x-values on your own to see that this is true).

Conclusion: $R_f = (-\infty,\ 0) \cup (0,\ \infty)$

Part #2: Find $f^{-1}(x)$

(1.) Solve $f(x) = \dfrac{1}{x+2}$ for x: (recall: $y = f(x)$).

$$y = \frac{1}{x+2}$$

$$y(x+2) = 1$$

$$xy + 2y = 1$$

$$xy = 1 - 2y$$

$$x = \frac{1-2y}{y}$$

(2.) Interchange $y = x$

$$y = \frac{1-2x}{x} = g(x)$$

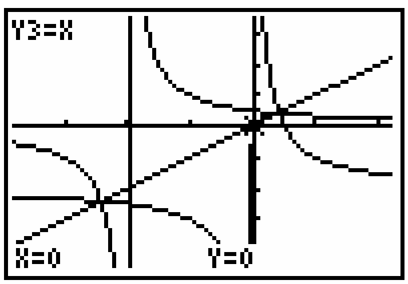
(3.) Verify $g(x) = f^{-1}(x)$ (This is left for you to verify on your own.)

Part #3: Determine if $f(x)$ is $1-1$

Using similar methods in *Part #1* above, graph $y = \frac{1-2x}{x}$ on your calculator and determine its domain and range. We conclude that $D_{f^{-1}} = (-\infty,\ 0) \cup (0,\ \infty)$ and $R_{f^{-1}} = (-\infty, -2) \cup (-2, \infty)$. Therefore we conclude that $f(x) = \frac{1}{x+2}$ is 1-1.

Part #4: Graph: $f(x)$ and $f^{-1}(x)$

Choose an appropriate window until your graph is similar to the one shown.



All points of intersection for the two curves occur on $y = x$. The extra vertical line that may appear indicates the graph is undefined at $x = -2$.